

On finding fields and self-force in a gauge appropriate to separable wave equations

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Abstract

Gravitational waves from the inspiral of a stellar-size black hole to a supermassive black hole can be accurately approximated by a point particle moving in a Kerr background. This paper presents progress on finding the electromagnetic and gravitational field of a point particle in a black-hole spacetime and on computing the self-force in a “radiation gauge.” The gauge is chosen to allow one to compute the perturbed metric from a gauge-invariant component ψ_0 (or ψ_4) of the Weyl tensor and follows earlier work by Chrzanowski and Cohen and Kegeles (we correct a minor, but propagating, error in the Cohen-Kegeles formalism). The electromagnetic field tensor and vector potential of a static point charge and the perturbed gravitational field of a static point mass in a Schwarzschild geometry are found, surprisingly, to have closed-form expressions. The gravitational field of a static point charge in the Schwarzschild background must have a strut, but ψ_0 and ψ_4 are smooth except at the particle, and one can find local radiation gauges for which the corresponding spin ± 2 parts of the perturbed metric are smooth. Finally a method for finding the renormalized self-force from the Teukolsky equation is presented. The method is related to the MiSaTaQuWa renormalization and to the Detweiler-Whiting construction of the singular field. It relies on the fact that the renormalized ψ_0 (or ψ_4) is a *sourcefree* solution to the Teukolsky equation; and one can therefore reconstruct a nonsingular renormalized metric in a radiation gauge.

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I. INTRODUCTION

Among the primary targets of the proposed space-based gravitational wave observatory, LISA, are waves from stellar black holes spiraling in to supermassive black holes in galactic centers. Because the ratio m/M is small, the orbits and gravitational waves from these binary systems can be described to high accuracy by a perturbative expansion with m/M as the small parameter. In addition, because tidal forces are small, one can model the system as a point mass orbiting a black hole. To zeroth order in m/M , the orbit is simply a geodesic of the black-hole spacetime. To first order, the particle feels a self-force, whose dissipative part is the radiation-reaction force that drives the inspiral. The self-force also has a conservative part that alters the phase of the orbit and of the emitted radiation [24, 25].

Because the dissipative part of the self-force is antisymmetric under the change from ingoing to outgoing radiation – from advanced to retarded fields, it can be computed from the half-retarded – half-advanced Green’s function. Because this Green’s function is sourcefree, it is regular at the particle. Approximating the self-force by its dissipative part is an adiabatic approximation [17, 18, 19], and several computations of orbits and waveforms have recently been carried out [8, 9, 12, 28].

Including the conservative part of the self force is a more difficult problem, because it arises from a field (the half-retarded + half-advanced part of the field) that is singular at the particle. One must renormalize the perturbed metric near the particle by subtracting off its singular part, a field singular at the position of the particle that does not itself contribute to the self force. The MiSaTaQuWa prescription for this subtraction, given by Mino, Sasaki and Tanaka[20] and subsequently in a particularly clear form by Quinn and Wald [31], is well understood in a Lorenz (deDonder, harmonic) gauge. A Lorenz gauge, however, is not well-adapted to the Kerr geometry: Instead of the decoupled, separable Teukolsky equation that simplifies black-hole perturbation theory, one must solve a system of ten coupled partial differential equations.

We report here the beginning of a program to compute the self-force in a gauge appropriate to the separable wave equation, using a formalism due to Chrzanowski [6] and to Cohen and Kegeles [4, 13] (henceforth CCK) to reconstruct the metric, in what is termed a *radiation gauge*, from either of the gauge-invariant components ψ_0 or ψ_4 that satisfy the Teukolsky

equation. Cohen and Kegeles clarified and made minor corrections to Chrzanowski's work,¹ and Wald gave a more concise derivation [33]. Subsequent work on inverting the differential equations that give the Hertz potential is reported in [15] and [23], and a first explicit vacuum reconstruction of the metric in a radiation gauge is given by Yunes and González [35]. In the present paper, we give a first explicit reconstruction of the metric of a point particle in a radiation gauge. And we outline how one can use a version of the MiSaTaQuWa renormalization due to Detweiler and Whiting [7] to renormalize the gauge invariant component ψ_0 (or ψ_4) of the perturbed Weyl tensor and then to reconstruct the renormalized metric in a radiation gauge from the renormalized Weyl tensor. (A different renormalization procedure, also based on a radiation gauge, is given by Barack and Ori [3].)

Cohen and Kegeles show that the reconstruction of the vector potential from the spin-weight ± 1 component ϕ_0 or ϕ_2 of the electromagnetic field tensor $F^{\alpha\beta}$ is closely analogous to the reconstruction of the perturbed metric from ψ_0 or ψ_4 , and we begin with this simpler example. After a mathematical introduction, we obtain in Sect. III closed form expressions for all components (ϕ_0, ϕ_1 and ϕ_2) of the electromagnetic field tensor of a static charge in a Schwarzschild background, a problem initially solved by Copson[5]. From ϕ_2 , we obtain the vector potential in a radiation gauge. A radiation gauge exists only where there are no sources. We observe that, in a region R , in order to obtain from a Hertz potential a smooth vector potential in a radiation gauge, spheres in R enclose no charge; we conjecture that this is generally true. One can obtain a smooth vector potential in a radiation gauge by adding an $l = 0$ part of the field in another gauge. (And, at least in a Schwarzschild background, one can alternatively add an $l = 0$ field in a radiation gauge that does not arise from a Hertz potential.)

We next (in Sect. IV) obtained a closed-form expression for the components ψ_0 and ψ_4 of the perturbed Weyl tensor of a static point mass in a Schwarzschild background. There is no consistent solution to the perturbed Einstein equation whose source is a static point mass: The mass must be supported. The support, however, can be a strut that does

¹ (Eq. (1.7) in [6] appears incorrectly to identify the Hertz potential with a component of the perturbed Weyl tensor, but the identification is made correctly later in the paper, apart from a missing complex conjugation; Eqs. (1.3)-(1.5) seem incorrectly to imply that radiation gauges can be used when matter is present.) The equations are correctly given by Cohen and Kegeles, apart from the factor-of-two correction we make here.

not contribute to the spin-weight ± 2 components of the Weyl tensor. In reconstructing the metric, we again find that the perturbed metric in a radiation gauge that arises from a Hertz potential is smooth only if spheres enclose no perturbed mass. Again one can obtain a smooth perturbed metric in a radiation gauge by adding an $l = 0$ part of the field in another gauge. This is part of the freedom one has to add an algebraically special perturbation without changing the spin-weight ± 2 part of the field on which the radiation gauge is based. Regardless of the choice of gauge, one cannot simultaneously make the metric smooth everywhere inside and outside the radius of the point mass. To obtain a smooth metric in a radiation gauge for the spin ± 2 part of the metric, one can add a strut by adding an algebraically special perturbation – in this case a perturbed C-metric having a nonzero deficit angle along an axis through the particle.

The strut is a feature of our static example that does not appear in an inspiral problem, and we conclude our sample reconstructions by considering a point mass in flat space, in which we choose a null tetrad that is again not centered at the perturbing mass. The formalism is now very close to that of the point charge in a Schwarzschild background, in which the only singularity in the perturbed metric outside the particle is pure gauge and can be removed by adding an $l = 0$ perturbation in a different gauge.

Finally, in Sect. VI, we note that the form given by Detweiler and Whiting for the singular part of the perturbed metric can be used to find the singular part of ψ_0 (or ψ_4). The renormalized Weyl tensor component, $\psi^{\text{ren}} = \psi_0^{\text{ret}} - \psi_0^{\text{sing}}$, satisfies the *sourcefree* Teukolsky equation. As a result, one can find a nonsingular renormalized metric perturbation in a radiation gauge. The motion of a particle is then given to first order in m/M by the requirement that it move on a geodesic of the renormalized metric.

II. MATHEMATICAL PRELIMINARIES

Greek letters early in the alphabet α, β, \dots will be abstract spacetime indices; letters μ, ν, \dots will be concrete indices, labeling components along the tetrad defined in Eq. (2) below. We adopt the $+- --$ signature of Newman and Penrose (NP)[21], writing the Schwarzschild metric in the form

$$ds^2 = \frac{\Delta}{r^2} dt^2 - \frac{r^2}{\Delta} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad \text{where } \Delta = r^2 - 2Mr. \quad (1)$$

We primarily use the Kinnersley tetrad[14], $\{e_\mu\}$ (numbered following NP notation),

$$\begin{aligned} e_1^\alpha \equiv l^\alpha &= \frac{r^2}{\Delta} t^\alpha + r^\alpha & e_2^\alpha \equiv n^\alpha &= \frac{1}{2} t^\alpha - \frac{1}{2} \frac{\Delta}{r^2} r^\alpha \\ e_3^\alpha \equiv m^\alpha &= \frac{1}{\sqrt{2}} (\hat{\theta}^\alpha + i \hat{\phi}^\alpha) & e_4^\alpha \equiv \bar{m}^\alpha, \end{aligned} \quad (2)$$

where we denote by t^α and r^α the vectors ∂_t and ∂_r and by $\hat{\theta}^\alpha$ and $\hat{\phi}^\alpha$ the unit vectors $\frac{1}{r} \partial_\theta$ and $\frac{1}{r \sin \theta} \partial_\phi$. The derivative operators associated with $l^\alpha, n^\alpha, m^\alpha$ and \bar{m}^α are, as usual denoted by D, Δ, δ and $\bar{\delta}$, respectively, but with a boldface Δ to distinguish this symbol from $\Delta = r^2 - 2Mr$.

In terms of the nonzero spin coefficients of the Kinnersley tetrad,

$$\rho = -\frac{1}{r} \quad \beta = -\alpha = \frac{\cot \theta}{2\sqrt{2}r} \quad \gamma = \frac{M}{2r^2} \quad \mu = -\frac{1}{2r} \frac{\Delta}{r^2}, \quad (3)$$

the corresponding nonzero Christoffel symbols have the form

$$\begin{aligned} \Gamma_{12}^1 &= 2\gamma & \Gamma_{22}^2 &= -2\gamma & \Gamma_{33}^3 &= 2\beta & \Gamma_{43}^1 &= \mu \\ \Gamma_{13}^3 &= -\rho & \Gamma_{23}^3 &= \mu & \Gamma_{34}^1 &= \mu & \Gamma_{43}^2 &= -\rho \\ \Gamma_{14}^4 &= -\rho & \Gamma_{24}^4 &= \mu & \Gamma_{34}^2 &= -\rho & \Gamma_{43}^4 &= -2\beta \\ & & & & \Gamma_{34}^3 &= -2\beta & \Gamma_{44}^4 &= 2\beta. \end{aligned}$$

Here, for example, $e_2^\beta \nabla_\beta e_1^\alpha =: \Gamma_{12}^\mu e_\mu^\alpha$.

The electromagnetic field $F_{\alpha\beta}$ has independent complex components,

$$\phi_0 = F_{\alpha\beta} l^\alpha m^\beta \quad (4)$$

$$\phi_1 = \frac{1}{2} F_{\alpha\beta} (l^\alpha n^\beta - m^\alpha \bar{m}^\beta) \quad (5)$$

$$\phi_2 = F_{\alpha\beta} \bar{m}^\alpha n^\beta, \quad (6)$$

with spin-weights 1, 0, and -1, respectively, where each occurrence of m^α (\bar{m}^α) contributes 1 (-1) to the spin-weight. Similarly, the Weyl tensor $C_{\alpha\beta\gamma\delta}$ has independent components

$$\begin{aligned} \Psi_0 &= -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta \\ \Psi_1 &= -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta \\ \Psi_2 &= -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta \\ \Psi_3 &= -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta \\ \Psi_4 &= -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta, \end{aligned} \quad (7)$$

with spin-weights 2, 1, 0, -1 and -2 , respectively.

In the Schwarzschild geometry, only Ψ_2 is nonzero, and it has value

$$\Psi_2 = -\frac{M}{r^3}. \quad (8)$$

We define the perturbation $h_{\alpha\beta}$ of a background metric $g_{\alpha\beta}$ by considering a family of metrics $g_{\alpha\beta}(\zeta)$, and writing

$$h_{\alpha\beta} = \left. \frac{d}{d\zeta} g_{\alpha\beta} \right|_{\zeta=0}. \quad (9)$$

The corresponding components of the perturbed Weyl tensor along the unperturbed tetrad will be denoted by the lower-case symbols, ψ_0, \dots, ψ_4 : For example,

$$\psi_3 = - \left. \frac{d}{d\zeta} C_{\alpha\beta\gamma\delta} \right|_{\zeta=0} l^\alpha n^\beta \bar{m}^\gamma n^\delta. \quad (10)$$

Tensor components of the Kinnersley tetrad with spin-weight s have angular behavior given by spin-weighted spherical harmonics, ${}_s Y_{lm}(\theta, \phi)$, when the tensor belongs to an l, m representation of the rotation group [11]. To define ${}_s Y_{lm}$, one first introduces operators \eth (edth) and $\bar{\eth}$ that respectively raise and lower by 1 the spin-weight of a quantity η of spin-weight s :

$$\eth \eta = -(\sin \theta)^s (\partial_\theta + i \csc \theta \partial_\phi) (\sin \theta)^{-s} \eta \quad (11)$$

$$= -(\partial_\theta + i \csc \theta \partial_\phi - s \cot \theta) \eta, \quad (12)$$

$$\bar{\eth} \eta = -(\sin \theta)^{-s} (\partial_\theta - i \csc \theta \partial_\phi) (\sin \theta)^s \eta \quad (13)$$

$$= -(\partial_\theta - i \csc \theta \partial_\phi + s \cot \theta) \eta. \quad (14)$$

Then, for each value of s , the spin-weighted spherical harmonics are a complete set of orthonormal functions on the 2-sphere, given by

$${}_s Y_{l,m} = \begin{cases} [(l-s)!/(l+s)!]^{1/2} \eth^s Y_{lm}, & 0 \leq s \leq l, \\ (-1)^s [(l+s)!/(l-s)!]^{1/2} \bar{\eth}^{-s} Y_{lm}, & -l \leq s \leq 0. \end{cases} \quad (15)$$

TABLE I: Source function

ψ	s	T
ϕ_0	1	$\delta J_1 - (D - 3\rho)J_3$
$\rho^{-2}\phi_2$	-1	$\rho^{-2} \{(\Delta + 3\mu)J_4 - \bar{\delta}J_2\}$
ψ_0	2	$2(\delta - 2\beta)[(D - 2\rho)T_{13} - \delta T_{11}] +$ $2(D - 5\rho)[(\delta - 2\beta)T_{13} - (D - \rho)T_{33}]$
$\rho^{-4}\psi_4$	-2	$2\rho^{-4}(\Delta + 2\gamma + 5\mu)[(\bar{\delta} - 2\beta)T_{24} - (\Delta + \mu)T_{44}] +$ $2\rho^{-4}(\bar{\delta} - 2\beta)[\Delta + 2\gamma + 2\mu)T_{24} - \bar{\delta}T_{22}]$

They satisfy the identities,

$${}_s Y_{lm}^* = (-1)^{m+s} {}_{-s} Y_{lm}, \quad (16)$$

$$\eth {}_s Y_{lm} = [(l-s)(l+s+1)]^{1/2} {}_{s+1} Y_{lm}, \quad (17)$$

$$\bar{\eth} {}_s Y_{lm} = -[(l+s)(l-s+1)]^{1/2} {}_{s-1} Y_{lm}, \quad (18)$$

$$\eth \bar{\eth} {}_s Y_{lm} = -(l-s)(l+s+1) {}_s Y_{lm}, \quad (19)$$

$$\sum_{l=|s|}^{\infty} \sum_{m=-1}^l {}_s Y_{lm}^*(\theta', \phi') {}_s Y_{lm}(\theta, \phi) = \delta(\cos \theta' - \cos \theta) \delta(\phi' - \phi), \quad (20)$$

$$\int d\Omega {}_s Y_{lm}^* {}_s Y_{l'm'} = \delta_{ll'} \delta_{mm'}. \quad (21)$$

The spin ± 1 components ϕ_0 and ϕ_2 of the electromagnetic field and the spin ± 2 components ψ_0 and ψ_4 of the perturbed Weyl tensor satisfy decoupled wave equations, namely the Bardeen-Press equation [1] and its electromagnetic analog. These are the $a=0$, spin ± 1 and spin ± 2 cases of the Teukolsky equation, and they have the form

$$\left[\frac{r^2}{\Delta} \partial_t^2 - 2s \left(\frac{M}{\Delta} - \frac{1}{r} \right) \partial_t + \mathbb{L} \right] \psi = 4\pi T, \quad (22)$$

with

$$\mathbb{L} := -\frac{\Delta^{-s}}{r^2} \frac{\partial}{\partial r} \left(\Delta^{s+1} \frac{\partial}{\partial r} \right) - \frac{1}{r^2} \eth \bar{\eth}, \quad (23)$$

where ψ is any of the functions listed in the first column of Table I and T is the corresponding source listed in the third column.

Static Green's function

To compute the fields of static sources, we use the time-independent forms of these equations,

$$\mathbb{L}\psi = 4\pi T. \quad (24)$$

Let x and x' denote points of a $t = \text{constant}$ hypersurface of the Schwarzschild geometry. We find below that the source terms for a static particle are related by the operator $\bar{\partial}$ to a point source of the form $\delta^3(x, x')$, normalized by

$$1 = \int dV' \delta^3(x, x') = \int d^3x' \sqrt{{}^3g(x')} \delta^3(x, x'). \quad (25)$$

In Schwarzschild coordinates, with $\sqrt{{}^3g} = r^3 \Delta^{-1/2} \sin \theta$, we have

$$\delta^3(x, x') = r^{-3} \Delta^{1/2} \delta(r - r') \delta(\cos \theta - \cos \theta') \delta(\phi - \phi'). \quad (26)$$

Denote by $G(x, x')$ the Green's function satisfying Eq. (24) with this source,

$$\mathbb{L}G(x, x') = -4\pi \delta^3(x, x'). \quad (27)$$

A solution $\psi(x)$ to Eq. (24) then has the form

$$\psi(x) = \int G(x, x') T(x') dV'. \quad (28)$$

We construct $G(x, x')$ from solutions to the homogenous equation $\mathbb{L}\psi = 0$. From Eq. (23), these have the form $R(r)S(\theta, \phi)$, with

$$\bar{\partial}\partial S = \lambda S, \quad \frac{\Delta^{-s}}{r^2} \frac{\partial}{\partial r} \left(\Delta^{s+1} \frac{\partial R}{\partial r} \right) - \frac{\lambda}{r^2} R = 0. \quad (29)$$

From Eq. (19), the solutions to the angular equation are the spin-weighted spherical harmonics with eigenvalues $\lambda = -(l - s)(l + s + 1)$. The radial equation then takes the form

$$\frac{\Delta^{-s}}{r^2} \frac{\partial}{\partial r} \left(\Delta^{s+1} \frac{\partial R}{\partial r} \right) + \frac{1}{r^2} (l - s)(l + s + 1) R = 0. \quad (30)$$

Its solutions are given in terms of the associated Legendre polynomials by

$$R(r) = \frac{a}{\Delta^{s/2}} P_l^s \left(\frac{\mathbf{r}}{M} \right) + \frac{b}{\Delta^{s/2}} Q_l^s \left(\frac{\mathbf{r}}{M} \right), \quad (31)$$

where a and b are arbitrary constants and $\mathbf{r} = r - M$.

Using the Wronskian

$$W \{P_l^s(z), Q_l^s(z)\} = \frac{(-1)^s (l + s)!}{(1 - z^2)(l - s)!}, \quad (32)$$

for integral values of l and s , we obtain the Wronskian of the two radial solutions,

$$W \left\{ \frac{1}{\Delta^{s/2}} P_l^s \left(\frac{\mathbf{r}}{M} \right), \frac{1}{\Delta^{s/2}} Q_l^s \left(\frac{\mathbf{r}}{M} \right) \right\} = (-1)^{s+1} \frac{M^2}{\Delta^{s+1}} \frac{(l+s)!}{(l-s)!}. \quad (33)$$

Using this relation and the completeness and orthogonality relations for the ${}_s Y_{lm}$ to compute for each l, m the discontinuity in Eq. (27) across $r = r'$, we find for the Green's function the form

$$G(x, x') = (-1)^s \frac{4\pi}{M} \left(\frac{\Delta'}{\Delta} \right)^{s/2} \sum_{l=|s|}^{\infty} \sum_{m=-l}^l \frac{(l-|s|)!}{(l+|s|)!} P_l^{|s|} \left(\frac{\mathbf{r}_{<}}{M} \right) Q_l^{|s|} \left(\frac{\mathbf{r}_{>}}{M} \right) {}_s Y_{lm}(\theta, \phi) {}_s Y_{lm}^*(\theta', \phi'), \quad (34)$$

where $\mathbf{r}_{<} = \min(r, r') - M$ and $\mathbf{r}_{>} = \max(r, r') - M$.

A. Tetrads smooth at $\theta = 0$ and $\theta = \pi$ and a criterion for smoothness of a tensor field.

The Kinnersley tetrad is singular when $\theta = 0$ and $\theta = \pi$. In order to disentangle (1) the singularity arising from the choice of tetrad from (2) a singularity arising from the use of the radiation gauge and (3) a physical singularity associated with a static particle on a Schwarzschild background, we will use, in addition to the Kinnersley tetrad, a closely related tetrad, $\{\mathbf{e}_\mu^+\} = \{l^\alpha, n^\alpha, m_+^\alpha = e^{i\phi} m^\alpha, \bar{m}_+^\alpha = e^{-i\phi} \bar{m}^\alpha\}$, that is smooth everywhere except $\theta = \pi$. (By *smooth* we always mean C^∞). One can similarly replace m^α by $e^{-i\phi} m^\alpha$ to obtain a null tetrad $\{\mathbf{e}_\mu^-\}$, smooth everywhere except $\theta = 0$. Because there is no continuous vector field nonzero everywhere on a 2-sphere, and because m^α must be tangent to the symmetry spheres if it is orthogonal to the principal null directions, no null tetrad based on the principal null directions is smooth everywhere.

After showing that $e^{\pm i\phi} m^\alpha$ is smooth near $\theta = 0$ ($\theta = \pi$), we will find a simple criterion for smoothness of a tensor field near $\theta = 0$ and $\theta = \pi$ that involves only its components in the Kinnersley tetrad.

To see that $e^{i\phi} m^\alpha$ is smooth near $\theta = 0$, we introduce a chart $\{t, X, Y, Z\}$ that is smooth on the spacetime exterior to the horizon: With

$$X = r \sin \theta \cos \phi, \quad Y = r \sin \theta \sin \phi, \quad Z = r \cos \theta, \quad (35)$$

the Schwarzschild metric has the form

$$ds^2 = \left(1 - \frac{2M}{r} \right) dt^2 - dX^2 - dY^2 - dZ^2 - \frac{2M}{r^2(r-2M)} (XdX + YdY + ZdZ)^2,$$

smooth for $r > 2M$. Then $e^{i\phi}m^\alpha$ is smooth, because its components in this cartesian chart are smooth. We have

$$m'^Z = e^{i\phi}m^Z = e^{i\phi}m^\theta\partial_\theta Z = -\frac{1}{\sqrt{2}}\frac{X+iY}{r}, \quad (36)$$

$$m'^X = e^{i\phi}m^X = e^{i\phi}\frac{1}{\sqrt{2}r}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right)X = \frac{1}{\sqrt{2}}\left[1 - \frac{X}{r}(X+iY)\frac{r-Z}{X^2+Y^2}\right], \quad (37)$$

$$m'^Y = e^{i\phi}m^Y = e^{i\phi}\frac{1}{\sqrt{2}r}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right)Y = \frac{1}{\sqrt{2}}\left[i - \frac{Y}{r}(X+iY)\frac{r-Z}{X^2+Y^2}\right]. \quad (38)$$

Finally, $\frac{r-Z}{X^2+Y^2}$ is smooth everywhere except the negative Z-axis, because it is analytic in $\{X, Y, Z\}$: This is obvious for $(X, Y) \neq 0$. To show that it is true for $(X, Y) = 0$, $Z > 0$, write $w = (X/r)^2 + (Y/r)^2$. Then, for $Z > 0$,

$$F(w) \equiv r\frac{r-Z}{X^2+Y^2} = \frac{1-\sqrt{1-w}}{w} = \sum a_n w^n, \quad a_n = \frac{1}{2}\frac{(1-\frac{1}{2})(2-\frac{1}{2})}{2}\frac{(2-\frac{1}{2})}{3}\dots\frac{(n-\frac{1}{2})}{n+1} < 1, \quad (39)$$

implying $F(w)$ analytic for $|w| < 1$. Thus $\{l^\alpha, n^\alpha, m_+^\alpha, \bar{m}_+^\alpha\}$ is a smooth (and analytic) basis for $r > 2M$, except on the negative Z-axis. Similarly, $\{l^\alpha, n^\alpha, m_-^\alpha, \bar{m}_-^\alpha\}$ is a smooth (and analytic) basis for $r > 2M$, except on the positive Z-axis.

A tensor field is smooth if and only if its components in a smooth basis are smooth. Thus a tensor field is smooth on the positive (negative) Z-axis if and only if its components along the basis $\{\mathbf{e}_\mu^\pm\}$ are smooth. Now a function $f e^{im\phi}$ (with f independent of ϕ) is smooth at $\theta = 0$ (and $f e^{-im\phi}$ is smooth at $\theta = \pi$) if and only if $f = g \sin^m \theta$, with g smooth. If f is a spin-weight m component of a tensor in the Kinnersley basis, then $f e^{\pm im\phi}$ is the corresponding component in the basis $\{\mathbf{e}_\mu^\pm\}$. Replacing $\sin^m \theta$ by θ^m for θ near 0 and $(\pi - \theta)^m$ for θ near π , we obtain the following criterion for the smoothness of a tensor:

Proposition. A tensor is smooth near $\theta = 0$ if and only if its components with spin-weight m in a Kinnersley tetrad have the form $\theta^m g$, with g smooth near $\theta = 0$; and it is smooth near $\theta = \pi$ if and only if its components in a Kinnersley tetrad have the form $(\pi - \theta)^m g$, with g a function smooth near $\theta = \pi$.

In the next three sections, we will use the CCK formalism to write the vector potential A_α in terms of a scalar Φ and to write the perturbed metric $h_{\alpha\beta}$ in terms of a scalar Ψ , a Hertz potential. We show in Appendix A that the Hertz potentials Ψ and Φ associated with a Kinnersley tetrad are related by phases, $e^{-2i\phi}$ and $e^{-i\phi}$ respectively, to the Hertz potential

associated with a smooth basis. It follows that $h_{\alpha\beta}$ and A_α are smooth if $\Psi/\sin^2\theta$ and $\Phi/\sin\theta$ are smooth. (As noted below, the converse is not true: A singular Hertz potential can yield a smooth h or A .)

III. STATIC CHARGE IN A SCHWARZSCHILD SPACETIME

A. Outline

We can now compute the electromagnetic field of a static point charge in a Schwarzschild background, finding its complex components (4-6) along the Kinnersley tetrad. We first use the decoupled equation (22) to find ϕ_2 , obtaining a closed-form expression. Then, following Cohen and Kegeles [4, 13], we construct a potential Φ , in terms of which we compute a vector potential A_α and the remaining components ϕ_0 and ϕ_1 of the field.² Because ϕ_2 has spin-weight -1, it has no $l = 0$ part, and the monopole part of the field – a change in the charge of the black hole – must be determined separately. In this formalism, the potential Φ is obtained from ϕ_2 by solving a second-order differential equation, and freedom to add electric and magnetic charge to the black hole can be identified with one of the constants of integration. Because the gauges for A_α introduced by Cohen and Kegeles exist globally only for a source-free solution, we find local gauges that are singular on different parts of the axis of symmetry. Explicit gauge transformations are found relating the vector potential to that found by Copson in 1928 [5].

B. Computing ϕ_2 from the decoupled wave equation

Consider a static charge e at a point with radial coordinate r_0 , outside the horizon of a Schwarzschild black hole; and choose the $\theta = 0$ axis to pass through the charge. The charge has current 4-vector $J^\alpha = \rho_e u^\alpha$, with 4-velocity $u^\alpha = \frac{r}{\Delta^{1/2}} t^\alpha$ and with charge density given

² Cohen and Kegeles denote by the same symbol Ψ Hertz potentials both for the electromagnetic field and for the perturbed Weyl tensor. To avoid this ambiguity, we denote by Φ the electromagnetic Hertz potential, retaining Ψ for the gravitational Hertz potential: $\Phi = \Psi(\text{Cohen-Kegeles, electromagnetic})$, $\Psi = \Psi(\text{Cohen-Kegeles, gravitational})$.

in the notation of Eq. (25) by

$$\rho_e(x) = e\delta^3(x, x_0) = e\frac{\Delta^{1/2}}{r^3}\delta(r - r_0)\delta(\cos\theta - 1)\delta(\phi). \quad (40)$$

The component ϕ_2 satisfies the wave equation (22) with spin $s = -1$. Note first that in the source term corresponding to $s = -1$ in Table I the angular component J_4 vanishes. The only contribution to T is then from the component $J_2 \equiv J_\alpha n^\alpha$,

$$T = -\frac{1}{\rho^2}\bar{\delta}J_2. \quad (41)$$

Acting on the spin-weight 0 component J_2 , the operator $\bar{\delta}$ is $-\frac{1}{r\sqrt{2}}\bar{\delta}$, implying

$$T = \frac{r}{\sqrt{2}}\bar{\delta}J_2. \quad (42)$$

From Eq. (2), we have

$$J_2 = \left(\rho_e \frac{r}{\Delta^{1/2}} t_\alpha\right) \left(\frac{1}{2} t^\alpha\right) = -\frac{1}{2} \frac{\Delta^{1/2}}{r} \rho_e \quad (43)$$

$$= -\frac{1}{2} e \frac{\Delta}{r^4} \delta(r - r_0) \delta(\cos\theta - 1) \delta(\phi). \quad (44)$$

We next decompose the source into spin-weighted spherical harmonics, using the completeness relation (20):

$$J_2 = -\frac{1}{2} e \frac{\Delta}{r^4} \delta(r - r_0) \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}(\theta, \phi) Y_{lm}^*(0, 0), \quad (45)$$

$$T(x) = -\frac{e\Delta}{2\sqrt{2}r^3} \delta(r - r_0) \bar{\delta} \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}(\theta, \phi) Y_{lm}^*(0, 0) \quad (46)$$

$$= -\frac{e\Delta}{2\sqrt{2}r^3} \delta(r - r_0) \sum_{l=1}^{\infty} \sum_{m=-l}^l [l(l+1)]^{1/2} {}_{-1}Y_{lm}(\theta, \phi) Y_{lm}^*(0, 0). \quad (47)$$

It is now straightforward to compute ϕ_2 from the Green's function $G(x, x')$ of Eq. (34), using orthonormality (21) of the ${}_sY_{lm}$. We have

$$\rho^{-2}\phi_2 = \int G(x, x')T(x')\sqrt{-g}d^3x' \quad (48)$$

$$= \int \left[-\frac{4\pi}{M} \left(\frac{\Delta}{\Delta'} \right)^{1/2} \sum_{l=1}^{\infty} \sum_{m=-l}^l \frac{1}{l(l+1)} P_l^1 \left(\frac{\mathbf{r}_{<}}{M} \right) Q_l^1 \left(\frac{\mathbf{r}_{>}}{M} \right) {}_{-1}Y_{lm}(\theta, \phi) {}_{-1}Y_{lm}^*(\theta', \phi') \right] \\ \times \left[-\frac{e\Delta'}{2\sqrt{2}r'^3} \delta(r' - r_0) \sum_{n=1}^{\infty} \sum_{p=-n}^n [n(n+1)]^{1/2} {}_{-1}Y_{np}(\theta', \phi') Y_{np}^*(0, 0) \right] r'^2 dr' d\Omega' \quad (49)$$

$$= \frac{4\pi e(\Delta\Delta_0)^{1/2}}{2\sqrt{2}Mr_0} \sum_{l=1}^{\infty} \sum_{m=-l}^l \frac{1}{[l(l+1)]^{1/2}} P_l^1 \left(\frac{\mathbf{r}_{<}}{M} \right) Q_l^1 \left(\frac{\mathbf{r}_{>}}{M} \right) {}_{-1}Y_{lm}(\theta, \phi) Y_{lm}^*(0, 0), \quad (50)$$

where $\mathbf{r}_{<} \equiv \min(r, r_0) - M$, $\mathbf{r}_{>} \equiv \max(r, r_0) - M$ and $\Delta_0 = r_0^2 - 2Mr_0$. Summing this series (details in the Appendix) yields the simple closed-form expression,

$$\phi_2 = e \frac{\Delta_0}{2\sqrt{2}r_0} \frac{\Delta \sin \theta}{r^2 R^3}, \quad (51)$$

where

$$R(r, \theta) := (\mathbf{r}^2 + \mathbf{r}_0^2 - 2\mathbf{r}\mathbf{r}_0 \cos \theta - M^2 \sin^2 \theta)^{1/2}, \quad (52)$$

$$\mathbf{r} = r - M \quad \text{and} \quad \mathbf{r}_0 = r_0 - M. \quad (53)$$

An entirely analogous computation of ϕ_0 , from Eq. (22) with spin-weight 1, yields the expression

$$\phi_0 = -e \frac{\Delta_0}{\sqrt{2}r_0} \frac{\sin \theta}{R^3}. \quad (54)$$

Our aim, however, is to parallel the Cohen-Kegeles treatment of gravitational perturbations, in which the metric and the remaining components of the Weyl tensor are constructed from a Hertz potential Ψ . Here the vector potential A_α plays the role of the metric, and A_α , ϕ_0 and ϕ_1 are constructed from an analogous Cohen-Kegeles Hertz potential Φ .

It is worth pointing out here that, although the electromagnetic field is smooth (C^∞) outside the particle, its components ϕ_0 and ϕ_2 are not smooth scalars.

C. The Cohen-Kegeles formalism for electromagnetism

We begin with a review of the Cohen-Kegeles [4, 13] formalism for electromagnetic fields on type D spacetimes. Underlying the formalism is the following relation between a vector potential satisfying the sourcefree Maxwell equation and a scalar $\bar{\Phi}$:

Proposition 1. On a vacuum type-D spacetime, let A_α be a smooth vector field of the form

$$A_\alpha = -m_\alpha(D + 2\epsilon + \rho)\Phi + l_\alpha(\delta + 2\beta + \tau)\Phi + c.c., \quad (55)$$

where Φ satisfies the sourcefree Teukolsky equation for $s = -1$ (for ϕ_2),

$$[(\Delta + \gamma - \bar{\gamma} + \bar{\mu})(D + 2\epsilon + \rho) - (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\delta + 2\beta + \tau)]\Phi = 0. \quad (56)$$

Then A_α satisfies the sourcefree Maxwell equations, and the complex components of the corresponding field tensor, $F_{\alpha\beta} := \nabla_\beta A_\alpha - \nabla_\alpha A_\beta$, are given by

$$\phi_0 = -(D - \epsilon + \bar{\epsilon} - \bar{\rho})(D + 2\bar{\epsilon} + \bar{\rho})\bar{\Phi} \quad (57)$$

$$\begin{aligned} 2\phi_1 = & -[(D + \epsilon + \bar{\epsilon} + \rho - \bar{\rho})(\bar{\delta} + 2\bar{\beta} + \bar{\tau}) \\ & + (\bar{\delta} - \alpha + \bar{\beta} - \pi - \bar{\tau})(D + 2\bar{\epsilon} + \bar{\rho})]\bar{\Phi} \end{aligned} \quad (58)$$

$$\phi_2 = -[(\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\bar{\delta} + 2\bar{\beta} + \bar{\tau}) - \lambda(D + 2\bar{\epsilon} + \bar{\rho})]\bar{\Phi}. \quad (59)$$

The vector potential given here is in what Chrzanowski and several subsequent authors call the *ingoing radiation gauge* (IRG), in which $A_\alpha l^\alpha = 0$. The terminology, however, is misleading: An *outgoing* radiation field in a transverse gauge has, near future null infinity, wave-vector l^α orthogonal to A_α (Chrzanowski incorrectly claims in Sect. IV of [6] that this is true of an ingoing field near past null infinity).

An identical proposition holds for the corresponding “ORG” gauge, in which the roles of the outgoing and ingoing null vectors, l^α and n^α , are exchanged. Because of the incorrect identification of “outgoing” and “ingoing” in the literature we will simply use the term *radiation gauges*.

Two comments on what the proposition does *not* imply: Note first that the component ϕ_2 of the electromagnetic field constructed from A_α is not the function Φ , although both Φ and ϕ_0 satisfy the same sourcefree Teukolsky equation. Second, the fact that ϕ_2 , say, satisfies the sourcefree Teukolsky equation does not guarantee that Φ satisfies the sourcefree Teukolsky equation. Freedom arising from constants of integration must be used to obtain a Φ that does so.

The expressions for A_α and for the components of the field tensor in terms of Φ are much simpler in a Schwarzschild geometry. In this case, all spin coefficients are real and most vanish:

$$\epsilon = \tau = \lambda = \pi = \alpha + \beta = 0. \quad (60)$$

Eqs. (55-59) then become

$$A_\alpha = -m_\alpha(D + \rho)\Phi + l_\alpha(\delta + 2\beta)\Phi + c.c. \quad (61)$$

$$[(\Delta + \mu)(D + \rho) - \bar{\delta}(\delta + 2\beta)]\Phi = 0 \quad (62)$$

$$\phi_0 = -D^2\bar{\Phi} \quad (63)$$

$$\phi_1 = -D(\bar{\delta} + 2\beta)\bar{\Phi} \quad (64)$$

$$\phi_2 = -\bar{\delta}(\bar{\delta} + 2\beta)\bar{\Phi}. \quad (65)$$

For future reference, we note that each operator involving δ or $\bar{\delta}$ is equal to $\bar{\partial}$ or $\bar{\partial}$ up to factors of $-r\sqrt{2}$. In particular, Eq. (65) for ϕ_2 can be written in the form,

$$\phi_2 = -\frac{1}{2r^2}\bar{\partial}^2\bar{\Phi}. \quad (66)$$

D. The scalar potential $\bar{\Phi}$

We now invert the CK relation (66) to find $\bar{\Phi}$ in terms of the field component ϕ_2 of a static charge, using the closed-form expression (51) for ϕ_2 . From Eq. (13) and the axisymmetry of ϕ_2 , we see that the leftmost $\bar{\partial}$ is just $-\partial_\theta$. Integrating with respect to θ , we have

$$\bar{\partial}\bar{\Phi} = \frac{e}{\sqrt{2}r_0}\Delta \int \left[\frac{\sin\theta}{R^3(r,\theta)} \right] d\theta \quad (67)$$

$$= \frac{e}{\sqrt{2}r_0} \left\{ \frac{M^2 \cos\theta - \mathbf{r}\mathbf{r}_0}{R} + a(r) \right\}, \quad (68)$$

with $a(r)$ an arbitry function of r . Similarly, after writing $\bar{\partial}\bar{\Phi} = -(\sin\theta)^{-1}\partial_\theta(\sin\theta\bar{\Phi})$, a second integration with respect to θ yields

$$\bar{\Phi} = \frac{e}{\sqrt{2}r_0\sin\theta} \int \left[\frac{M^2 \cos\theta - \mathbf{r}\mathbf{r}_0}{R} + a(r) \right] d\cos\theta \quad (69)$$

$$= \frac{e}{\sqrt{2}r_0\sin\theta} [R + a(r)\cos\theta + b(r)], \quad (70)$$

with $b(r)$ again arbitrary.

Eq. (70) has the form of a particular + a homogeneous solution to Eq. (66),

$$\bar{\Phi} = \bar{\Phi}_P + \bar{\Phi}_H, \quad (71)$$

$$\bar{\Phi}_P = \frac{e}{\sqrt{2}r_0} \frac{R}{\sin\theta}, \quad (72)$$

$$\bar{\Phi}_H = \frac{e}{\sqrt{2}r_0} \frac{a(r)\cos\theta + b(r)}{\sin\theta}. \quad (73)$$

The particular solution $\bar{\Phi}_P$ already satisfies the sourcefree Teukolsky equation. The homogeneous solution, $\bar{\Phi}_H$, however, does so if and only if $a''(r) = b''(r) = 0$. With this restriction, the solutions $\bar{\Phi}_H$ constitute a 4-parameter set, specified by

$$a(r) = a_0 + a_1 r, \quad b(r) = b_0 + b_1 r. \quad (74)$$

By construction, a and b encode no information about ϕ_2 . Because we expect ϕ_2 to carry all information about the $l \geq 1$ part of the field, the value of ϕ_0 should similarly be independent of a and b . This follows directly from Eqs. (63) and (74), which together imply that $\phi_0[\bar{\Phi}_H] = 0$. Then a and b can carry information only about the monopole field. As we show in the next section, the parameter a_0 corresponds to adding charge to the black hole. The remaining parameters a_1, b_0 and b_1 correspond simply to gauge transformations (with b_1 the trivial gauge transformation, altering neither the field nor the vector potential).

The potential $\bar{\Phi}$ is in general singular on the z axis because of an overall factor of $1/\sin\theta$. This singularity is carried to the vector potential as well, because the angular pieces of equation (61) contain only radial derivatives. We will see that one can choose a_1 and b_0 to make A_α smooth on the axis for $r > r_0$ or for $r < r_0$, but not both.

E. Completion of the solution

Fields corresponding to $\bar{\Phi}_H$

The vector potential $A_\alpha[\bar{\Phi}_H]$, associated with the homogeneous part of the potential

$$\bar{\Phi}_H = \frac{e}{\sqrt{2} r_0 \sin\theta} [(a_0 + a_1 r) \cos\theta + (b_0 + b_1 r)], \quad (75)$$

has in the radiation gauge the value

$$A_\alpha[\bar{\Phi}_H] = \frac{e}{2r_0} \left[-a_0 \left(\frac{1}{r} l_\alpha - \sqrt{2} \frac{\cot\theta}{r} m_\alpha \right) - a_1 l_\alpha + b_0 \frac{\sqrt{2}}{r \sin\theta} m_\alpha \right] + c.c. \quad (76)$$

Using the relations

$$l_\alpha = \nabla_\alpha u, \quad \frac{1}{r \sin\theta} m_\alpha = -\frac{1}{\sqrt{2}} \nabla_\alpha \left(\log \tan \frac{\theta}{2} + i\phi \right), \quad (77)$$

with u the null coordinate $u = t - r - 2M \ln(r/2M - 1)$, we have

$$A_\alpha[\bar{\Phi}_H] = -a_0 \frac{e}{2r_0 r} \left(l_\alpha - \sqrt{2} \cot\theta m_\alpha \right) + c.c. + \nabla_\alpha \Lambda, \quad (78)$$

where the gauge scalar Λ is given by

$$\Lambda = \frac{e}{2r_0} \left[a_1 u - b_0 \left(\ln \tan \frac{\theta}{2} + i\phi \right) \right] + c.c. \quad (79)$$

Thus, as claimed, the vector potentials associated with a_1 and b_0 are pure gauge, and that associated with b_1 vanishes.

The identification of a_0 with a change in the black hole's charge (electric and magnetic) follows from the form of the corresponding field tensor: By a test charge q on a Schwarzschild black hole is meant a radial electric field, with $F_{\alpha\beta} \hat{t}^\alpha \hat{r}^\beta = \frac{q}{r^2}$. Equivalently, only the spin-0 part of the field is nonzero, and it is real, with value $\phi_1 = -\frac{q}{2r^2}$. From Eqs. (63-65), the field associated with $\bar{\Phi}_H$ for a_0 real is the field of a black-hole test charge of magnitude $Q = \frac{ea_0}{r_0}$:

$$\phi_0[\bar{\Phi}_H] = \phi_2[\bar{\Phi}_H] = 0, \quad (80)$$

$$\phi_1[\bar{\Phi}_H] = -\frac{ea_0}{2r_0} \frac{1}{r^2}. \quad (81)$$

Similarly, a test magnetic charge g on a Schwarzschild black has the meaning of a radial magnetic field with $F_{\alpha\beta} \hat{\theta}^\alpha \hat{\phi}^\beta = \frac{g}{r^2}$. Equivalently, only ϕ_1 is nonzero, and it is imaginary, with value $\phi_1 = \frac{g}{2r^2}$. And the field associated with $\bar{\Phi}_H$ for a_0 imaginary is the field of a test magnetic charge of magnitude $g = \frac{e(\text{Im } a_0)}{r_0}$.

If we replace a_0 in $\bar{\Phi}_H$ by $Q \equiv r_0 a_0 / e$, we can summarize the last paragraph as follows: The electromagnetic field associated with $\bar{\Phi} = \frac{Q}{\sqrt{2}} \cot \theta$ is a monopole field with electric charge $\text{Re } Q$ and magnetic charge $\text{Im } (-Q)$.

Fields corresponding to $\bar{\Phi}_P$

The vector potential corresponding to $\bar{\Phi}_P$ has the value

$$A_\alpha[\bar{\Phi}_P] = -\frac{e}{\sqrt{2}r_0 \sin \theta} \left[\frac{\mathbf{r} - \mathbf{r}_0 \cos \theta}{R} - \frac{R}{r} \right] (m_\alpha + \bar{m}_\alpha) + \frac{e}{r_0 r R} [\mathbf{r} \mathbf{r}_0 - M^2 \cos \theta] l_\alpha. \quad (82)$$

The corresponding components of the field, given by Eqs. (63-65), are

$$\phi_0 = -e \frac{\Delta_0}{\sqrt{2} r_0} \frac{\sin \theta}{R^3} \quad (83)$$

$$\phi_1[\bar{\Phi}_P] = \partial_r \left\{ \frac{e}{2r_0 r} \frac{M^2 \cos \theta - \mathbf{r} \mathbf{r}_0}{R} \right\} \quad (84)$$

$$\phi_2 = e \frac{\Delta_0}{2\sqrt{2} r_0} \frac{\Delta \sin \theta}{r^2 R^3} \quad (85)$$

As we now show, the black-hole charge for this field is nonzero. To find the field of a point charge for which the black hole has zero charge, one must add to $\phi_1[\bar{\Phi}_P]$ a multiple of $\phi_1[\bar{\Phi}_H]$ that has black-hole charge opposite to that of our particular solution. The total charge of the particular solution (black-hole charge + point charge) can be found by evaluating $4\pi Q = \int_S \vec{E} \cdot d\vec{S} = \int_S 2\phi_1 dS$ over any 2-sphere enclosing the black hole and particle. We can, for simplicity, take the sphere to be at spatial infinity, writing

$$Q = \frac{1}{4\pi} \lim_{r \rightarrow \infty} \int_{S_r} \partial_r \left\{ \frac{e}{2r_0 r} \frac{M^2 \cos \theta - \mathbf{r} \cdot \mathbf{r}_0}{R} \right\} r^2 d\Omega \quad (86)$$

$$= e \left(1 - \frac{M}{r_0} \right). \quad (87)$$

Thus the particular linearized solution associated with $\bar{\Phi}_P$ describes a particle of charge e outside a black hole of charge

$$Q_{\text{BH}} = -e \frac{M}{r_0}. \quad (88)$$

We will see in the next section that any homogeneous solution $\bar{\Phi}_H$ that makes the vector potential smooth for $r < r_0$ has a black hole charge that exactly cancels the black hole charge of the particular solution: A radiation gauge is smooth for $2M < r < r_0$ only if spheres with $2M < r < r_0$ have no enclosed charge. Similarly, we will see that a radiation gauge is smooth for $r > r_0$ only if the total charge (black hole + particle) vanishes: Spheres with $r > r_0$ enclose no charge.

Parameter choices for solutions smooth on the axis of symmetry

As we showed in Appendix A, a *sufficient* condition for A_α to be smooth is that the Hertz potential $\bar{\Phi}$ is smooth; and $\bar{\Phi}$ of the form given by Eqs. (71) – (74) is smooth if and only if $\bar{\Phi} = O(\theta)$ near $\theta = 0$ and $\bar{\Phi} = O(\pi - \theta)$ near $\theta = \pi$. Smoothness of $\bar{\Phi}$, however, is not a *necessary* condition for smoothness of A_α , because $\bar{\Phi}$ involves a parameter b_1 that corresponds to a vanishing vector potential. By omitting conditions on b_1 , we obtain necessary and sufficient conditions for smoothness of A_α . (It is slightly more efficient to find the conditions from $\bar{\Phi}$, but we will also obtain them directly from the components of A_α as a check.)

From Eqs. (72) and (75), we have

$$\bar{\Phi} = \bar{\Phi}_P + \bar{\Phi}_H = \frac{e}{\sqrt{2} r_0 \sin \theta} [R + (a_0 + a_1 r) \cos \theta + b_0 + b_1 r]. \quad (89)$$

Smoothness on the $\theta = 0$ part of the axis (the part of the axis that contains the particle) is enforced by different conditions below and above the particle's position at $r = r_0$. Formally,

the difference arises from the fact that the quantity $R = (\mathfrak{r}^2 + \mathfrak{r}_0^2 - 2\mathfrak{r}\mathfrak{r}_0 \cos \theta - M^2 \sin^2 \theta)^{1/2}$ in the numerator of the above expression for $\bar{\Phi}$ has the value $R(\theta = 0) = |r - r_0|$. Then $\bar{\Phi}$ is smooth at $\theta = 0$ if and only if $a_0 + a_1 r + b_0 + b_1 r = -|r - r_0|$, or

$$a_0 + b_0 = -r_0, \quad a_1 + b_1 = 1, \quad r < r_0 \quad (90a)$$

$$a_0 + b_0 = r_0, \quad a_1 + b_1 = -1, \quad r > r_0. \quad (90b)$$

Similarly, from the value $R(\theta = \pi) = r + r_0 - 2M$, it follows that $\bar{\Phi}$ is smooth at $\theta = \pi$ if and only if

$$a_0 - b_0 = r_0 - 2M. \quad (91)$$

The conditions on a_0 and b_0 in Eqs. (90) are inconsistent, implying that no choice of parameters yields a radiation gauge smooth both below and above the particle. One can at most choose parameters that satisfy condition (91) and one of the two conditions (90). These choices yield a vector potential A_α

(I) smooth everywhere except $\theta = 0$, $r > r_0$, or

(II) smooth everywhere except $\theta = 0$, $r < r_0$.

The implied choice of parameters for each case is given in Table II. Note that any choice

TABLE II: Singularity location for two choices of parameters

Parameter	Singularity at	Singularity at
	$\theta = 0; r \geq r_0$	$\theta = 0; r \leq r_0$
a_0	$-M$	$r_0 - M$
b_0	$-(r_0 - M)$	M

of the parameter a_1 is permitted, because one can choose the trivial parameter b_1 to satisfy the second condition without altering the value A_α .

In case (1) the vector potential is, in particular, smooth in the exterior Schwarzschild geometry for all $r < r_0$. The parameter value $a_0 = M$ corresponds to a black hole charge eM/r_0 that exactly cancels the black hole charge (88) of the particular solution. In case (2) the vector potential is smooth for all $r > r_0$, and the parameter value $a_0 = r_0 - M$ corresponds to a black hole charge $-e(1 - M/r_0)$ that cancels the total charge of the particular solution: The full solution has a black hole with charge $-e$ and total charge 0. Thus as noted above,

a Hertz potential $\bar{\Phi}$ yields a smooth radiation gauge on a sphere only if the total electric and magnetic flux through the sphere vanish.

This is not surprising. The radiation gauges are designed for sourcefree solutions and are constructed from the spin ± 1 parts of the field - in our case from parts of the field with no $l = 0$ part. For a point-charge with no charge on the black hole, we can find regular radiation gauges outside and inside the charge for the $l \geq 1$ part of the field and can add to them the vector potential in another gauge for the Coulomb $l = 0$ field.

It makes sense to use a radiation gauge to describe the part of the field with spin greater than 0 and to add an $l = 0$ part in another gauge. In particular, by adding the solution describing a perturbation of the black hole's charge in a Coulomb gauge, namely

$$\begin{aligned} A_t &= \frac{Q}{r}, \\ A_\alpha &= A_t \nabla_\alpha t = \frac{Q}{r} \left(\frac{1}{2} l_\alpha + \frac{r^2}{\Delta} n_\alpha \right), \end{aligned} \quad (92)$$

we obtain a solution with arbitrary black hole charge, whose gauge singularity is restricted to lie on the $\theta = 0$ ray, either above or below the particle.³

The vector potentials corresponding to cases (I) (singular only for $\theta = 0, r \geq r_0$) and (II) (singular only for $\theta = 0, r \leq r_0$) are, respectively,

$$\begin{aligned} A_\alpha^{(I)} &= \frac{e}{r_0 r} \left(\frac{\mathbf{r} \mathbf{r}_0 - M^2 \sin^2 \theta}{R} + M \right) l_\alpha \\ &\quad + \frac{e}{\sqrt{2} r_0 \sin \theta} \left(\frac{R - \mathbf{r}_0 - M \cos \theta}{r} - \frac{\mathbf{r} - \mathbf{r}_0 \cos \theta}{R} \right) (m_\alpha + \bar{m}_\alpha), \end{aligned} \quad (93a)$$

$$\begin{aligned} A_\alpha &= \frac{e}{r_0 r} \left(\frac{\mathbf{r} \mathbf{r}_0 - M^2 \sin^2 \theta}{R} - \mathbf{r}_0 \right) l_\alpha \\ &\quad + \frac{e}{\sqrt{2} r_0 \sin \theta} \left(\frac{R + \mathbf{r}_0 + M \cos \theta}{r} - \frac{\mathbf{r} - \mathbf{r}_0 \cos \theta}{R} \right) (m_\alpha + \bar{m}_\alpha). \end{aligned} \quad (93b)$$

Here we have set the parameter a_1 to zero.

As we have noted, Eqs. (93a) and (93b) are physically different solutions, with the former having an uncharged black hole, the latter a black hole with charge $-e$. Adding to the vector potential (93b) the $l = 0$ (Coulomb) perturbation (92) that cancels the charge $-e$ on the

³ Although one cannot obtain the $l = 0$ vector potential in a radiation gauge by using a Hertz potential of the form (89), the vector potential *can* be expressed in a radiation gauge: Writing $f = Q \ln \left(\frac{r}{2M} - 1 \right)$, we have $\tilde{A}_\alpha \equiv A_\alpha - \nabla_\alpha f = \frac{Q}{r} n_\alpha$, implying $\tilde{A}_\alpha l^\alpha = 0$.

black hole, we obtain a vector potential that is gauge related to that of Eq. (93a) and is smooth except on the axis below the particle:

$$A_\alpha^{(II)} = \frac{e}{r_0 r} \left(\frac{\mathbf{r}\mathbf{r}_0 - M^2 \sin^2 \theta}{R} + M \right) l_\alpha + \frac{er}{\Delta} n_\alpha + \frac{e}{\sqrt{2} r_0 \sin \theta} \left(\frac{R + \mathbf{r}_0 + M \cos \theta}{r} - \frac{\mathbf{r} - \mathbf{r}_0 \cos \theta}{R} \right) (m_\alpha + \bar{m}_\alpha). \quad (94)$$

Finally, as a check, we directly verify the smoothness conditions governing a_0 and b_0 in (90) and (91), using the components of $A_\alpha[\bar{\Phi}] = A_\alpha[\bar{\Phi}_P] + A_\alpha[\bar{\Phi}_H]$ given by Eqs. (76) and (82). Because A_2 is already smooth for arbitrary values of the parameters, A_α will be smooth if and only if A_4 is $O(\theta)$ near $\theta = 0$ and $O(\pi - \theta)$ near $\theta = \pi$. Near $\theta = 0$, we have

$$A_4[\bar{\Phi}] = -\frac{e}{\sqrt{2} r_0 r} \frac{a_0 + b_0 + r_0}{\theta} + O(\theta), \quad r < r_0 \quad (95a)$$

$$A_4[\bar{\Phi}] = \frac{e}{\sqrt{2} r_0 r} \frac{a_0 + b_0 - r_0}{\theta} + O(\theta), \quad r > r_0. \quad (95b)$$

Near the axis of symmetry on the side of the black hole opposite to the particle ($\theta = \pi$) the corresponding expansion of A_α has the form

$$A_4 = -\frac{e}{\sqrt{2} r_0 r} \frac{-a_0 + b_0 + r_0 - 2M}{\pi - \theta} + O[(\pi - \theta)]. \quad (96)$$

The conditions on a_0 and b_0 in Eqs. (90) and (91) are exactly conditions that the coefficients of θ^{-1} and $(\pi - \theta)^{-1}$ vanish.

We can explicitly verify that the vector potentials $A_\alpha^{(I)}$ and $A_\alpha^{(II)}$ we have obtained for a point charge outside an uncharged black hole are related by a gauge transformation to the solution obtained by Copson. In particular, using the gauge function

$$\Lambda(t, r, \theta, \phi) = -e \ln \left[\frac{\mathbf{r}_0 \cos \theta - \mathbf{r} + R}{\cos \theta + 1} \right] + \frac{eM}{r_0} \ln [2(\mathbf{r}_0 \cos \theta - \mathbf{r} + R)] + \frac{eM}{r_0} \ln \left[\frac{\Delta_0 - M^2 \sin^2 \theta + Mr - \mathbf{r}_0 r \cos \theta + (M(1 + \cos \theta) - r_0) R}{M(1 + \cos \theta) \sin^2 \theta} \right], \quad (97)$$

we find $A_\alpha^{(I)} + \nabla_\alpha \Lambda = A_t \nabla_\alpha t$, with

$$A_t = \frac{e}{rr_0} \left[\frac{\mathbf{r}\mathbf{r}_0 - M^2 \sin^2 \theta}{R} + M \right]. \quad (98)$$

IV. STATIC GRAVITATIONAL PERTURBATIONS

A. Outline

Notation. Because the components ψ_0, ψ_1, ψ_3 and ψ_4 of the Weyl tensor vanish in the background Schwarzschild spacetime, we use the same symbols to denote the components of the perturbed Weyl tensor. Because Ψ_2 is nonzero in the background spacetime, we denote the perturbed component by ψ_2 .

Outline of Computation. To compute the linearized gravitational field of a point mass in the Cohen-Kegeles formalism, we roughly parallel the calculation of the electromagnetic field of a point charge, but there are two primary differences. First, where the electromagnetic field tensor is gauge invariant, only the extreme spin components, ψ_0 and ψ_4 , of the perturbed Weyl tensor are gauge invariant. Second, the electromagnetic field of a static test charge makes sense, but a static test mass must be supported by something that itself contributes to the linearized gravitational field, and in our solution this is seen as a conical singularity (a deficit angle) along the axis of symmetry inside or outside $r = r_0$. Nevertheless, the extreme spin components of the perturbed Weyl tensor are smooth everywhere except at the position of the particle: The part of the perturbation with $|s| = 2$ does not know about the strut. As in the electromagnetic case, we obtain simple closed-form expressions for these gauge-invariant parts of the static perturbation.

We first use the decoupled equation (22) to find ψ_4 , again by taking angular derivatives of the static Green's function. Using a radiation gauge, we construct a potential Ψ , in terms of which we compute the components $h_{\alpha\beta}$ of the perturbed metric and the remaining components ψ_1, ψ_2 and ψ_3 of the field. In so doing, we correct of factor-of-two error in the Cohen-Kegeles formalism.

Because ψ_4 has spin-weight -2 , it has no $l = 0$ or $l = 1$ parts. The monopole part of the field corresponds to a change in the mass or area of the black hole, the dipole part to a change in its angular momentum and center of mass. As in the case of the electromagnetic charge, the change in the black hole's mass must be specified separately. The change in its center of mass appears as a gauge transformation. The potential Ψ is obtained from ψ_4 by solving a fourth-order differential equation, and the freedom to add charge and spin to the

black hole can be identified with two of the the constants of integration. We again find local gauges that are singular on different parts of the axis of symmetry (the singularities in these radiation gauges were identified by Barack and Ori[2]), with a residual conical singularity (the strut) that cannot be removed by a choice of gauge.

B. Computing ψ_4 from the decoupled wave equation

Consider a static point particle of mass \mathbf{m} at a point x_0 with coordinates $r = r_0$, $\theta = 0$. The particle has density

$$\rho_m(x) = \mathbf{m}\delta^3(x, x_0) \quad (99)$$

and stress-energy tensor

$$T^{\alpha\beta} = \rho_m u^\alpha u^\beta, \quad \text{with } u^\alpha = \frac{r}{\Delta^{1/2}} t^\alpha. \quad (100)$$

In the source term of the Bardeen-Press equation for $\rho^{-4}\psi_4$, given in Table I, only the component T_{22} is nonzero, and the source is simply

$$T = -2(\bar{\delta} - 2\beta)\bar{\delta}\rho^{-4}T_{22}, \quad (101)$$

with

$$T_{22} \equiv T_{\alpha\beta}n^\alpha n^\beta = \mathbf{m}\frac{\Delta^{3/2}}{4r^5}\delta(r - r_0)\delta(\cos\theta - 1)\delta(\phi). \quad (102)$$

By writing the angular derivatives in the source in terms of $\bar{\partial}$ and using equation (20), we can expand the source in spin-weighted spherical harmonics:

$$T(x) = -2(\bar{\delta} - 2\beta)\bar{\delta}\rho^{-4}T_{22} \quad (103)$$

$$= -r^2\bar{\partial}^2T_{22} \quad (104)$$

$$= -\mathbf{m}\frac{\Delta^{3/2}}{4r^3}\delta(r - r_0)\bar{\partial}^2\sum_{l=0}^{\infty}\sum_{m=-l}^l Y_{lm}(\theta, \phi)Y_{lm}^*(0, 0) \quad (105)$$

$$= -\mathbf{m}\frac{\Delta^{3/2}}{4r^3}\delta(r - r_0)\sum_{l=2}^{\infty}\sum_{m=-l}^l \left[\frac{(l+2)!}{(l-2)!}\right]^{1/2} {}_{-2}Y_{lm}(\theta, \phi)Y_{lm}^*(0, 0) \quad (106)$$

We can now obtain ψ_4 from the Green's function of Eq. (34). Eq. (28) for $\rho^{-4}\psi_4$ has the

form,

$$\begin{aligned}
\rho^{-4}\psi_4 &= \int G(x, x')T(x')\sqrt{-g}d^3x' \\
&= \int \left[\frac{4\pi\Delta}{M\Delta'} \sum_{l=2}^{\infty} \sum_{m=-l}^l \frac{(l-2)!}{(l+2)!} P_l^2\left(\frac{\mathbf{r}_{\leq}}{M}\right) Q_l^2\left(\frac{\mathbf{r}_{\geq}}{M}\right) {}_{-2}Y_{lm}(\theta, \phi) {}_{-2}Y_{lm}^*(\theta', \phi') \right] \\
&\quad \times \left[-\frac{\mathbf{m}\Delta'^{3/2}}{4r'^3} \delta(r' - r_0) \sum_{n=2}^{\infty} \sum_{p=-n}^n \left[\frac{(n+2)!}{(n-2)!} \right]^{1/2} {}_{-2}Y_{np}(\theta', \phi') Y_{np}^*(0, 0) \right] r'^2 dr' d\Omega'
\end{aligned} \tag{107}$$

and, using orthogonality of the spin-weighted harmonics, we have

$$\psi_4 = -\frac{\pi\mathbf{m}}{M} \frac{\Delta\Delta_0^{1/2}}{r^4 r_0} \sum_{l=2}^{\infty} \sum_{m=-l}^l \left[\frac{(l-2)!}{(l+2)!} \right]^{1/2} P_l^2\left(\frac{\mathbf{r}_{\leq}}{M}\right) Q_l^2\left(\frac{\mathbf{r}_{\geq}}{M}\right) {}_{-2}Y_{lm}(\theta, \phi) Y_{lm}^*(0, 0). \tag{109}$$

It is again possible to sum this series to obtain the closed-form expression (details in Appendix B),

$$\psi_4 = -\mathbf{m} \frac{3\Delta_0^{3/2}}{4r_0} \frac{\Delta^2 \sin^2 \theta}{r^4 R^5}. \tag{110}$$

C. The Cohen-Kegeles formalism for gravity

The formalism of Cohen and Kegeles [13] again relates ψ_4 to a scalar potential Ψ which we can then use to reconstruct the perturbed metric $h_{\alpha\beta}$. In each of the key relations, Eqs. (113-117) below, we correct an error that appears in the Cohen-Kegeles papers and has been repeated in the subsequent literature: The right hand side of each equation is reduced by the factor 1/2 from the corresponding Cohen-Kegeles expressions.

Proposition 2. On a vacuum type-D spacetime, let $h_{\alpha\beta}$ be a smooth tensor field of the form

$$\begin{aligned}
h_{\alpha\beta} &= -\{ l_{\alpha} l_{\beta} [(\bar{\delta} + \alpha + 3\bar{\beta} - \bar{\tau})(\bar{\delta} + 4\bar{\beta} + 3\bar{\tau}) - \lambda(D + 4\bar{\epsilon} + 3\bar{\rho})] \\
&\quad + \bar{m}_{\alpha} \bar{m}_{\beta} (D - \epsilon + 3\bar{\epsilon} - \bar{\rho})(D + 4\bar{\epsilon} + 3\bar{\rho}) \\
&\quad - l_{(\alpha} \bar{m}_{\beta)} [(D + \epsilon + 3\bar{\epsilon} + \rho - \bar{\rho})(\bar{\delta} + 4\bar{\beta} + 3\bar{\tau}) + \\
&\quad (\bar{\delta} - \alpha + 3\bar{\beta} - \pi - \bar{\tau})(D + 4\bar{\epsilon} + 3\bar{\rho})] \} \bar{\Psi} + c.c.,
\end{aligned} \tag{111}$$

where Ψ satisfies the sourcefree Teukolsky equation for $s = -2$ (for ψ_4),

$$[(\delta + 3\bar{\alpha} + \beta - \tau)(\bar{\delta} + 4\bar{\beta} + 3\bar{\tau}) - (\Delta - \gamma + 3\bar{\gamma} + \mu)(D + 4\bar{\epsilon} + 3\bar{\rho}) + 3\bar{\Psi}_2] \bar{\Psi} = 0. \tag{112}$$

Then $h_{\alpha\beta}$ satisfies the perturbed vacuum Einstein equation, and the complex components of the corresponding perturbed Weyl tensor are given by

$$\psi_0 = \frac{1}{2}(D - 3\epsilon + \bar{\epsilon} - \bar{\rho})(D - 2\epsilon + 2\bar{\epsilon} - \bar{\rho})(D - \epsilon + 3\bar{\epsilon} - \bar{\rho})(D + 4\bar{\epsilon} + 3\bar{\rho})\bar{\Psi} \quad (113)$$

$$\begin{aligned} \psi_1 = & \frac{1}{8}[(D - \epsilon + \bar{\epsilon} + \rho - \bar{\rho})(D + 2\bar{\epsilon} + \rho - \bar{\rho})(D + \epsilon + 3\bar{\epsilon} + \rho - \bar{\rho})(\bar{\delta} + 4\bar{\beta} + 3\bar{\tau}) \\ & + (D - \epsilon + \bar{\epsilon} + \rho - \bar{\rho})(D + 2\bar{\epsilon} + \rho - \bar{\rho})(\bar{\delta} - \alpha + 3\bar{\beta} - \pi - \bar{\tau})(D + 4\bar{\epsilon} + 3\bar{\rho}) \\ & + (D - \epsilon + \bar{\epsilon} + \rho - \bar{\rho})(\bar{\delta} - 2\alpha + 2\bar{\beta} - 2\pi - \bar{\tau})(D - \epsilon + 3\bar{\epsilon} - \bar{\rho})(D + 4\bar{\epsilon} + 3\bar{\rho}) \\ & + (\bar{\delta} - 3\alpha + \bar{\beta} - 3\pi - \bar{\tau})(D - 2\epsilon + 2\bar{\epsilon} - \bar{\rho})(D + \epsilon + 3\bar{\epsilon} - \bar{\rho})(D + 4\bar{\epsilon} + 3\bar{\rho})]\bar{\Psi} \quad (114) \end{aligned}$$

$$\begin{aligned} \psi_2 = & \frac{1}{12}[(D + \epsilon + \bar{\epsilon} + 2\rho - \bar{\rho})(D + 2\epsilon + 2\bar{\epsilon} + 2\rho - \bar{\rho})(\bar{\delta} + \alpha + 3\bar{\beta} - \bar{\tau})(\bar{\delta} + 4\bar{\beta} + 3\bar{\tau}) \\ & + (D + \epsilon + \bar{\epsilon} + 2\rho - \bar{\rho})(\bar{\delta} + 2\bar{\beta} - \pi - \bar{\tau})(D + \epsilon + 3\bar{\epsilon} + \rho - \bar{\rho})(\bar{\delta} + 4\bar{\beta} + 3\bar{\tau}) \\ & + (D + \epsilon + \bar{\epsilon} + 2\rho - \bar{\rho})(\bar{\delta} + 2\bar{\beta} - \pi - \bar{\tau})(\bar{\delta} - \alpha + 3\bar{\beta} - \pi - \bar{\tau})(D + 4\bar{\epsilon} + 3\bar{\rho}) \\ & + (\bar{\delta} - \alpha + \bar{\beta} - 2\pi - \bar{\tau})(\bar{\delta} - 2\alpha + 2\bar{\beta} - 2\pi - \bar{\tau})(D - \epsilon + 3\bar{\epsilon} - \bar{\rho})(D + 4\bar{\epsilon} + 3\bar{\rho}) \\ & + (\bar{\delta} - \alpha + \bar{\beta} - 2\pi - \bar{\tau})(D + 2\bar{\epsilon} + \rho - \bar{\rho})(\bar{\delta} - \alpha + 3\bar{\beta} - \pi - \bar{\tau})(D + 4\bar{\epsilon} + 3\bar{\rho}) \\ & + (\bar{\delta} - \alpha + \bar{\beta} - 2\pi - \bar{\tau})(D + 2\bar{\epsilon} + \rho - \bar{\rho})(D + \epsilon + 3\bar{\epsilon} + \rho - \bar{\rho})(\bar{\delta} + 4\bar{\beta} + 3\bar{\tau})]\bar{\Psi} \quad (115) \end{aligned}$$

$$\begin{aligned} \psi_3 = & \frac{1}{8}[(D + 3\epsilon + \bar{\epsilon} + 3\rho - \bar{\rho})(\bar{\delta} + 2\alpha + 2\bar{\beta} - \bar{\tau})(\bar{\delta} + \alpha + 3\bar{\beta} - \bar{\tau})(\bar{\delta} + 4\bar{\beta} + 3\bar{\tau}) \\ & + (\bar{\delta} + \alpha + \bar{\beta} - \pi - \bar{\tau})(D + 2\epsilon + 2\bar{\epsilon} + 2\rho - \bar{\rho})(\bar{\delta} + \alpha + 3\bar{\beta} - \bar{\tau})(\bar{\delta} + 4\bar{\beta} + 3\bar{\tau}) \\ & + (\bar{\delta} + \alpha + \bar{\beta} - \pi - \bar{\tau})(\bar{\delta} + 2\bar{\beta} - \pi - \bar{\tau})(D + \epsilon + 3\bar{\epsilon} + \rho - \bar{\rho})(\bar{\delta} + 4\bar{\beta} + 3\bar{\tau}) \\ & + (\bar{\delta} + \alpha + \bar{\beta} - \pi - \bar{\tau})(\bar{\delta} + 2\bar{\beta} - \pi - \bar{\tau})(\bar{\delta} - \alpha + 3\bar{\beta} - \pi - \bar{\tau})(D + 4\bar{\epsilon} + 3\bar{\rho})]\bar{\Psi} \quad (116) \end{aligned}$$

$$\begin{aligned} \psi_4 = & \frac{1}{2}\{(\bar{\delta} + 3\alpha + \bar{\beta} - \bar{\tau})(\bar{\delta} + 2\alpha + 2\bar{\beta} - \bar{\tau})(\bar{\delta} + \alpha + 3\bar{\beta} - \bar{\tau})(\bar{\delta} + 4\bar{\beta} + 3\bar{\tau})\bar{\Psi} \\ & + 3\Psi_2[\tau(\bar{\delta} + 4\alpha) - \rho(\Delta + 4\gamma) - \mu(D + 4\epsilon) + \pi(\delta + 4\beta) + 2\Psi_2]\Psi\} \quad (117) \end{aligned}$$

Note that the metric $h_{\alpha\beta}$ is tracefree and satisfies $h_{\alpha\beta}l^\beta = 0$. Because these are five real conditions on the components of $h_{\alpha\beta}$, and one has only four gauge degrees of freedom, one cannot find a radiation gauge for a generic metric. As Price, Shankar and Whiting show, however [26], such a gauge exists locally for vacuum perturbations of any type D vacuum spacetime. In particular a perturbation that changes the mass of a Kerr spacetime (with J and M nonzero for the background) cannot be written in a radiation gauge, but radiative perturbations and a perturbation that changes the angular momentum of Kerr can be.

As we show below (and as Price et al. found independently), however, one *can* express a perturbation that changes the mass of a Schwarzschild black hole in a radiation gauge. We

exhibit two examples acquired from a Hertz potential that are each singular on a ray. Although the Hertz potential does not yield a form of the mass perturbation that is nonsingular in the entire exterior Schwarzschild geometry, the family of radiation gauges includes such a perturbation, and we present it below. The Hertz potential formalism, appropriate for spin-weight greater than two, yields at least for $s = 0$ perturbations (for $l = 0$ perturbations of Schwarzschild), a restricted family of radiation gauges.

By exchanging the roles of the ingoing and outgoing null vectors one obtains analogous propositions for a radiation gauge in which $h_{\alpha\beta}n^\beta = 0$, $h = 0$.

These equations are much simpler on a Schwarzschild background. The form of $h_{\alpha\beta}$ is

$$h_{\alpha\beta} = -[l_\alpha l_\beta (\bar{\delta} + 2\beta)(\bar{\delta} + 4\beta) + \bar{m}_\alpha \bar{m}_\beta (D - \rho)(D + 3\rho) - 2l_{(\alpha} \bar{m}_{\beta)} (D + \rho)(\bar{\delta} + 4\beta)] \bar{\Psi} + c.c., \quad (118)$$

where

$$\{(\delta - 2\beta)(\bar{\delta} + 4\beta) - (\Delta + 2\gamma + \mu)(D + 3\rho) + 3\Psi_2\} \bar{\Psi} = 0. \quad (119)$$

The expressions for the complex components of the Weyl tensor become

$$\psi_0 = \frac{1}{2} D^4 \bar{\Psi} \quad (120)$$

$$\psi_1 = \frac{1}{2} D^3 (\bar{\delta} + 4\beta) \bar{\Psi} \quad (121)$$

$$\psi_2 = \frac{1}{2} D^2 (\bar{\delta} + 2\beta)(\bar{\delta} + 4\beta) \bar{\Psi} \quad (122)$$

$$\psi_3 = \frac{1}{2} D \bar{\delta} (\bar{\delta} + 2\beta)(\bar{\delta} + 4\beta) \bar{\Psi} \quad (123)$$

$$\psi_4 = \frac{1}{2} (\bar{\delta} - 2\beta) \bar{\delta} (\bar{\delta} + 2\beta)(\bar{\delta} + 4\beta) \bar{\Psi} - \frac{3}{2} \Psi_2 [\mu D + \rho(\Delta + 4\gamma) - 2\Psi_2] \Psi. \quad (124)$$

In the next section, we solve Eq. (124) to find the scalar potential Ψ from the closed-form expression (110) for ψ_4 . In the second line of Eq. (124), the expression in brackets becomes simply $-\frac{1}{2r} \partial_t$, when one substitutes the Schwarzschild values of the operators and spin coefficients (3), and the value $-2M/r^3$ of the unperturbed Ψ_2 . In our static case, the second line then vanishes, leaving a set of angular derivatives of $\bar{\Psi}$. Replacing ψ_4 on the left side by its value (110), we have

$$-\frac{3\mathfrak{m}\Delta_0^{3/2}}{4r_0}\frac{\Delta^2\sin^2\theta}{r^4R^5} = \frac{1}{2}(\bar{\delta}-2\beta)\bar{\delta}(\bar{\delta}+2\beta)(\bar{\delta}+4\beta)\bar{\Psi} \quad (125)$$

$$= \frac{1}{8r^4}\bar{\delta}^4\bar{\Psi}, \quad (126)$$

or

$$\bar{\delta}^4\bar{\Psi} = -\frac{6\mathfrak{m}\Delta_0^{3/2}}{r_0}\frac{\Delta^2\sin^2\theta}{R^5}. \quad (127)$$

D. The scalar potential Ψ

We can now begin the integration of Eq. (127) for Ψ . Because ψ_4 has spin-weight -2 and $\bar{\delta}$ lowers the spin-weight by 1, $\bar{\Psi}$ has spin-weight $+2$. From the definition (13) of $\bar{\delta}$, its action on the spin-weight -1 quantity $\bar{\delta}^3\bar{\Psi}$ is given by

$$\bar{\delta}^4\bar{\Psi} = -\sin\theta\partial_\theta\left\{\frac{1}{\sin\theta}\bar{\delta}^3\bar{\Psi}\right\}. \quad (128)$$

Integrating this equation with respect to θ , we have

$$\bar{\delta}^3\bar{\Psi} = \frac{6\mathfrak{m}\Delta^2\Delta_0^{3/2}}{r_0}\sin\theta\int\frac{\sin\theta d\theta}{R^5} \quad (129)$$

$$= -\frac{6\mathfrak{m}\Delta^2\Delta_0^{3/2}}{r_0}\sin\theta\int\frac{d\cos\theta}{(\mathfrak{r}^2+\mathfrak{r}_0^2-2\mathfrak{r}\mathfrak{r}_0\cos\theta-M^2\sin^2\theta)^{5/2}} \quad (130)$$

$$= \frac{2\mathfrak{m}}{r_0\Delta_0^{1/2}}\sin\theta\left\{\frac{M^2\cos\theta-\mathfrak{r}\mathfrak{r}_0}{R^3}[\mathfrak{r}^2\mathfrak{r}_0^2-3M^2(\mathfrak{r}^2+\mathfrak{r}_0^2)+3M^4+4\mathfrak{r}\mathfrak{r}_0M^2\cos\theta-2M^4\cos^2\theta]\right. \\ \left.+a(r)\right\}, \quad (131)$$

with $a(r)$ arbitrary.

Next, acting on the spin-weight 0 quantity $\bar{\delta}^2\Psi$, $\bar{\delta}$ has the form

$$\bar{\delta}^3\bar{\Psi} = -\partial_\theta\bar{\delta}^2\bar{\Psi}, \quad (132)$$

and we have

$$\bar{\delta}^2\bar{\Psi} = \frac{2\mathfrak{m}}{r_0\Delta_0^{1/2}}\int\left\{\frac{M^2\cos\theta-\mathfrak{r}\mathfrak{r}_0}{R^3}[\mathfrak{r}^2\mathfrak{r}_0^2-3M^2(\mathfrak{r}^2+\mathfrak{r}_0^2)+3M^4+4\mathfrak{r}\mathfrak{r}_0M^2\cos\theta-2M^4\cos^2\theta]\right. \\ \left.+a(r)\right\}d\cos\theta \quad (133)$$

$$= \frac{2\mathfrak{m}}{r_0\Delta_0^{1/2}}\left\{-\frac{\mathfrak{r}^2\mathfrak{r}_0^2+M^2(\mathfrak{r}^2+\mathfrak{r}_0^2)-M^4-4\mathfrak{r}\mathfrak{r}_0M^2\cos\theta+2M^4\cos^2\theta}{(\mathfrak{r}^2+\mathfrak{r}_0^2-2\mathfrak{r}\mathfrak{r}_0\cos\theta-M^2\sin^2\theta)^{1/2}}\right. \\ \left.+a(r)\cos\theta+b(r)\right\}, \quad (134)$$

with $b(r)$ arbitrary.

Continuing in this way, we have

$$\bar{\partial}^2 \bar{\Psi} = -\frac{1}{\sin \theta} \partial_\theta (\sin \theta \bar{\partial} \bar{\Psi}), \quad (135)$$

whose integration entails a third free function $c(r)$:

$$\begin{aligned} \sin \theta \bar{\partial} \bar{\Psi} &= \frac{2\mathfrak{m}}{r_0 \Delta_0^{1/2}} \int \left\{ -\frac{\mathfrak{r}^2 \mathfrak{r}_0^2 + M^2(\mathfrak{r}^2 + \mathfrak{r}_0^2) - M^4 - 4\mathfrak{r}\mathfrak{r}_0 M^2 \cos \theta + 2M^4 \cos^2 \theta}{R} \right. \\ &\quad \left. + a(r) \cos \theta + b(r) \right\} d \cos \theta \\ &= \frac{2\mathfrak{m}}{r_0 \Delta_0^{1/2}} \left\{ (\mathfrak{r}\mathfrak{r}_0 - M^2 \cos \theta) R + \frac{1}{2} a(r) \cos^2 \theta + b(r) \cos \theta + c(r) \right\}. \end{aligned} \quad (136)$$

The final integration of

$$\sin \theta \bar{\partial} \bar{\Psi} = -\frac{1}{\sin \theta} \partial_\theta (\sin^2 \theta \bar{\Psi}), \quad (137)$$

yields

$$\begin{aligned} \sin^2 \theta \bar{\Psi} &= \frac{2\mathfrak{m}}{r_0 \Delta_0^{1/2}} \int \left\{ (\mathfrak{r}\mathfrak{r}_0 - M^2 \cos \theta) R + \frac{1}{2} a(r) \cos^2 \theta + b(r) \cos \theta + c(r) \right\} d \cos \theta \\ &= \frac{2\mathfrak{m}}{r_0 \Delta_0^{1/2}} \left\{ -\frac{1}{3} R^3 + \frac{1}{6} a(r) \cos^3 \theta + \frac{1}{2} b(r) \cos^2 \theta + c(r) \cos \theta + d(r) \right\}. \end{aligned} \quad (138)$$

We thus find a remarkably simple particular solution $\bar{\Psi}_P$, together with a homogeneous solution $\bar{\Psi}_H$ involving four free functions, $a(r), b(r), c(r)$ and $d(r)$:

$$\bar{\Psi} = \bar{\Psi}_P + \bar{\Psi}_H \quad (139)$$

$$\bar{\Psi}_P = -\frac{2}{3} A \frac{R^3}{\sin^2 \theta} \quad (140)$$

$$\bar{\Psi}_H = 2A \frac{1}{\sin^2 \theta} \left\{ \frac{1}{6} a(r) \cos^3 \theta + \frac{1}{2} b(r) \cos^2 \theta + c(r) \cos \theta + d(r) \right\}, \quad (141)$$

where we denote by A a constant that appears repeatedly throughout this section,

$$A := \frac{\mathfrak{m}}{r_0 \Delta_0^{1/2}}. \quad (142)$$

Again the particular solution $\bar{\Psi}_P$ already satisfies the homogeneous Bardeen-Press equation for $s = 2$. Requiring that $\bar{\Psi}_H$ also be a solution restricts each free function to a polynomial whose form is given in Table III. Then the scalars $\bar{\Psi}$ of Eq. (139) form an eight-complex-parameter set of potentials that generate vacuum perturbations for which ψ_4 has the value (110), with $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ the eight complex parameters.

TABLE III: Free functions of the gravitational scalar potential

Function	Value
$a(r)$	$a_1 r^2 (r - 3M) + a_2$
$b(r)$	$b_1 r^2 + (r - M)b_2$
$c(r)$	$-\frac{1}{2}a_1(r - M)(r^2 + 4M^2) - \frac{1}{2}a_2 + c_2(r - M) + c_1 r^2$
$d(r)$	$\frac{1}{2}b_1 r^2 + \frac{1}{2}b_2 r + d_1 r^2 (r - 3M) + d_2$

Now each Ψ_H generates a vacuum perturbation for which the gauge invariant component ψ_4 vanishes. It follows that ψ_0 vanishes as well, and a theorem of Wald [32] restricts the perturbations associated with Ψ_H to combinations of perturbations that change the mass and spin of the black hole, perturb Schwarzschild to a C-Metric or to Kerr-NUT, or are pure gauge. We begin the next section by characterizing in this way the perturbation associated with each of the eight parameters of Ψ_H .

E. Completion of the solution

Fields corresponding to Ψ_H

We will need only real values of the parameters a_1, \dots, d_2 , but will also include complex values of a_2 , because $\text{Im}(a_2)$ corresponds to a perturbation taking Schwarzschild to Kerr, adding angular momentum to the Schwarzschild geometry.

We note first that the perturbed metric associated with a_1 is a C-metric perturbation. The components of the perturbed metric and Weyl tensor in the radiation gauge have the following forms:

$$h_{22} = -a_1 A 2(r - 3M) \cos \theta \quad (143a)$$

$$h_{23} = a_1 A \frac{4M^2(r - 2M) - r^3 \sin^2 \theta}{\sqrt{2} r^2 \sin \theta} \quad (143b)$$

$$h_{33} = a_1 A \frac{2M(r \sin^2 \theta - 4M) \cos \theta}{r \sin^2 \theta}, \quad (143c)$$

$$\psi_0 = \psi_2 = \psi_4 = 0 \quad (144a)$$

$$\psi_1 = a_1 A \frac{6\sqrt{2}M^3}{r^4 \sin \theta} \quad (144b)$$

$$\psi_3 = -a_1 A \frac{3M \sin \theta}{2\sqrt{2} r^2}. \quad (144c)$$

To verify that this is a perturbed C Metric, we refer in Wald [32] to the perturbed tetrad produced by what he calls \dot{p} . Using this perturbed tetrad to compute the perturbation in the metric $g_{\alpha\beta} = 2n_{(\mu}l_{\beta)} - 2m_{(\mu}\overline{m}_{\beta)}$, we calculate a perturbed metric. It is then straightforward to find a gauge transformation from Wald's gauge to our radiation gauge.

When a_2 is real, the perturbation associated with it is pure gauge: The nonzero components of the perturbed metric,

$$h_{22} = -a_2 A \frac{2 \cos \theta}{r^2} \quad (145a)$$

$$h_{23} = a_2 A \frac{\sqrt{2} \sin \theta}{r^2} \quad (145b)$$

$$h_{33} = 0, \quad (145c)$$

and Weyl tensor,

$$\psi_1 = -a_2 A \frac{3 \sin \theta}{\sqrt{2} r^4} \quad (146a)$$

$$\psi_2 = a_2 A \frac{3 \cos \theta}{r^4} \quad (146b)$$

$$\psi_3 = a_2 A \frac{3 \sin \theta}{2\sqrt{2} r^4}, \quad (146c)$$

are the components of $\mathcal{L}_\xi g_{\alpha\beta}$ and $\mathcal{L}_\xi C_{\alpha\beta\gamma\delta}$ for the gauge vector

$$\xi_1 = a_2 A \frac{\cos \theta}{M}, \quad (147a)$$

$$\xi_2 = -a_2 A \frac{(r + 2M) \cos \theta}{2Mr}, \quad (147b)$$

$$\xi_3 = -a_2 A \frac{\sin \theta}{\sqrt{2}M}. \quad (147c)$$

For imaginary a_2 , the perturbation is no longer pure gauge. Instead it describes a change in the angular momentum of the black hole, given by

$$\dot{J} = ia_2 A. \quad (148)$$

The perturbed metric has components

$$h_{23} = -i\sqrt{2}j\frac{\sin\theta}{r^2}, \quad (149)$$

$$h_{22} = h_{33} = 0. \quad (150)$$

We show that this is a perturbation to Kerr, written in a radiation gauge, by exhibiting a gauge transformation to a gauge associated with Boyer-Lindquist coordinates. Denote by $g_{\alpha\beta}^{\text{kerr}}(M, J)$ the Kerr metric of mass M and angular momentum J . In Boyer-Lindquist coordinates, a perturbation $h_{\alpha\beta}^{\text{kerr}} = \frac{d}{d\zeta}g_{\alpha\beta}^{\text{kerr}}[M, J(\zeta)]_{\zeta=0}$ (with $J(0) = 0$) has as its only nonzero component

$$h_{t\phi}^{\text{kerr}} = -2\frac{j}{r}\sin\theta; \quad (151)$$

the corresponding nonzero components along the Kinnersley tetrad are

$$h_{13}^{\text{kerr}} = 2h_{23}^{\text{kerr}} = -i\sqrt{2}j\frac{\sin\theta}{\Delta}. \quad (152)$$

With the gauge vector ξ^α whose nonzero components are

$$\xi^3 = -\xi^4 = \frac{i}{\sqrt{2}}\frac{j}{M}\left[1 + \frac{r}{2M}\ln\left(1 - \frac{2M}{r}\right)\right], \quad (153)$$

the perturbed metric $h_{\alpha\beta} = h_{\alpha\beta}^{\text{kerr}} + \mathcal{L}_\xi g_{\alpha\beta}$ is the metric of Eq. (150), as claimed.

The perturbation associated with b_1 is a change in the mass of the black hole of magnitude $\dot{M} = 3MAb_1$. In the radiation gauge, metric and Weyl tensor perturbations have components

$$h_{22} = -2b_1A \quad (154a)$$

$$h_{23} = 0 \quad (154b)$$

$$h_{33} = b_1A\frac{2(1 + \cos^2\theta)}{\sin^2\theta}, \quad (154c)$$

$$\psi_0 = \psi_1 = \psi_2 = \psi_3 = \psi_4 = 0. \quad (155a)$$

To verify that this is a perturbation of the Schwarzschild mass, we exhibit a gauge transformation to a Schwarzschild gauge, a gauge in which $h_{tt} = \frac{d}{d\zeta}\left(1 - \frac{2M(\zeta)}{r}\right) = -\frac{2\dot{M}}{r}$, $h_{rr} = -\frac{d}{d\zeta}\left[\left(1 - \frac{2M(\zeta)}{r}\right)^{-1}\right] = -\frac{2\dot{M}}{r}\frac{r^4}{\Delta^2}$, with all other coordinate components vanishing. The corresponding nonzero tetrad components of $h_{\alpha\beta}$ in this gauge are

$$h_{11} = -\frac{4\dot{M}}{r}\frac{r^4}{\Delta^2}, \quad h_{22} = -\frac{\dot{M}}{r}. \quad (156)$$

The gauge transformation that yields a perturbed metric of this form is generated by the vector ξ^α with components

$$\xi_1 = b_1 A \frac{r}{\Delta} [r^2 - 4Mr - 8M^2 - t(r - 2M) + 8M(r - 2M) \ln(r - 2M)] \quad (157a)$$

$$\xi_2 = b_1 A \frac{1}{2r} [3r^2 - 4Mr - 8M^2 - t(r - 2M) + 8M(r - 2M) \ln(r - 2M)] \quad (157b)$$

$$\xi_3 = -b_1 A \sqrt{2} r \cot \theta. \quad (157c)$$

The corresponding gauge perturbation is given by

$$(\mathcal{L}_\xi g)_{11} = b_1 A \frac{12Mr^3}{\Delta^2} \quad (158a)$$

$$(\mathcal{L}_\xi g)_{22} = -b_1 A \frac{2r - 3M}{r} \quad (158b)$$

$$(\mathcal{L}_\xi g)_{33} = b_1 A \frac{2(1 + \cos^2 \theta)}{\sin^2 \theta} \quad (158c)$$

Finally, the gauge-transformed metric obtained by subtracting (158) from (154) has the form (156), with $\dot{M} = 3MAb_1$, as claimed.

The perturbation associated with b_2 is again a change in the mass of the black hole, in this case with $\dot{M} = Ab_2$. In the radiation gauge, it has the form

$$h_{22} = -b_2 A \frac{2(r - M)}{r^2} \quad (159a)$$

$$h_{23} = -b_2 A \frac{\sqrt{2}(r - 2M) \cos \theta}{r^2 \sin \theta} \quad (159b)$$

$$h_{33} = b_2 A \frac{2(1 + \cos^2 \theta)}{r \sin^2 \theta}, \quad (159c)$$

$$\psi_0 = \psi_3 = \psi_4 = 0 \quad (160a)$$

$$\psi_1 = -b_2 A \frac{3\sqrt{2}M \cos \theta}{r^4 \sin \theta} \quad (160b)$$

$$\psi_2 = b_2 A \frac{r - 3M}{r^4}. \quad (160c)$$

A gauge transformation to the Schwarzschild gauge (156) is in this case generated by the gauge vector

$$\xi_1 = -b_2 A \frac{r}{\Delta} \left[r + 2M - 2(r - 2M) \ln \left(\frac{r - 2M}{\sin \theta} \right) \right] \quad (161a)$$

$$\xi_2 = b_2 A \frac{\Delta}{2r^2} \left[1 + 2 \ln \left(\frac{r - 2M}{\sin \theta} \right) \right] \quad (161b)$$

$$\xi_3 = -b_2 A \sqrt{2} \cot \theta, \quad (161c)$$

for which

$$(\mathcal{L}_\xi g)_{11} = b_2 A \frac{4r^3}{\Delta^2} \quad (162a)$$

$$(\mathcal{L}_\xi g)_{22} = -b_2 A \frac{\Delta}{r^3} \quad (162b)$$

$$(\mathcal{L}_\xi g)_{23} = -b_2 A \frac{\sqrt{2}(r-2M)\cos\theta}{r^2 \sin\theta} \quad (162c)$$

$$(\mathcal{L}_\xi g)_{33} = b_2 A \frac{2(1+\cos^2\theta)}{r \sin^2\theta}. \quad (162d)$$

Subtracting (162) from (159) yields the form (156), with $\dot{M} = Ab_2$.

The remaining perturbations are all pure gauge. For each parameter, we list below the components of the associated metrics, the components of the perturbed Weyl tensor, and the nonzero components of the gauge vector.

c_1 :

$$h_{22} = h_{23} = 0 \quad (163a)$$

$$h_{33} = c_1 A \frac{4\cos\theta}{\sin^2\theta}, \quad (163b)$$

$$\psi_0 = \psi_1 = \psi_2 = \psi_3 = \psi_4 = 0, \quad (164a)$$

$$\xi_1 = -c_1 A \frac{\sqrt{2}r}{\sin\theta}. \quad (165)$$

c_2 :

$$h_{22} = 0 \quad (166a)$$

$$h_{23} = -c_2 A \frac{\sqrt{2}(r-2M)}{r^2 \sin\theta} \quad (166b)$$

$$h_{33} = c_2 A \frac{4\cos\theta}{r \sin^2\theta}, \quad (166c)$$

$$\psi_0 = \psi_2 = \psi_3 = \psi_4 = 0 \quad (167a)$$

$$\psi_1 = -c_2 A \frac{3\sqrt{2}M}{r^4 \sin\theta}, \quad (167b)$$

$$\xi_1 = c_2 A 2 \ln\left(\cot\frac{\theta}{2}\right) \quad (168a)$$

$$\xi_2 = c_2 A \frac{\Delta}{r^2} \ln\left(\cot\frac{\theta}{2}\right) \quad (168b)$$

$$\xi_3 = -c_2 A \frac{\sqrt{2}}{\sin\theta}. \quad (168c)$$

d_1 :

$$h_{22} = h_{23} = 0 \quad (169a)$$

$$h_{33} = -d_1 A \frac{12M}{\sin^2 \theta}, \quad (169b)$$

$$\psi_0 = \psi_1 = \psi_2 = \psi_3 = \psi_4 = 0, \quad (170a)$$

$$\xi_3 = d_1 A 3\sqrt{2}Mr (\cot \theta + i\phi \sin \theta) \quad (171a)$$

$$\xi_4 = d_1 A 3\sqrt{2}Mr (\cot \theta - i\phi \sin \theta). \quad (171b)$$

d_2 :

$$h_{22} = h_{23} = h_{33} = 0, \quad (172)$$

$$\psi_0 = \psi_1 = \psi_2 = \psi_3 = \psi_4 = 0. \quad (173)$$

Fields corresponding to Ψ_P

The perturbed metric associated with Ψ_P has the nonzero components

$$\begin{aligned} h_{22} &= -(\bar{\delta} + 2\bar{\beta})(\bar{\delta} + 4\bar{\beta})\bar{\Psi}_P - (\delta + 2\beta)(\delta + 4\beta)\Psi_P \\ &= A \frac{2}{r^2 R} [(\mathbf{r}^2 + \mathbf{r}_0^2)M^2 + \mathbf{r}^2 \mathbf{r}_0^2 - M^4 - 4M^2 \mathbf{r} \mathbf{r}_0 \cos \theta + 2M^4 \cos^2 \theta] \end{aligned} \quad (174a)$$

$$\begin{aligned} h_{23} &= (D + \rho)(\delta + 4\beta)\Psi_P + (D + \rho)(\bar{\delta} + 4\bar{\beta})\bar{\Psi}_P \\ &= A \frac{\sqrt{2}}{r^2 R \sin \theta} \{ (2\mathbf{r} - M)r\mathbf{r}_0 M - (\mathbf{r} - M)\mathbf{r}_0^3 + (\mathbf{r} - 2M)(rM^2 + (\mathbf{r} - M)\mathbf{r}_0^2) \cos \theta \\ &\quad - 2(2\mathbf{r} - M)\mathbf{r}_0 M^2 \cos^2 \theta + 2M^4 \cos^3 \theta \} \end{aligned} \quad (174b)$$

$$\begin{aligned} h_{33} &= (D - \rho)(D + 3\rho)\bar{\Psi}_P + (D - \rho)(D + 3\rho)\Psi_P \\ &= -A \frac{2}{r R \sin^2 \theta} \{ \mathbf{r} \mathbf{r}_0^2 - M^2 \mathbf{r} - M(2\mathbf{r}^2 + \mathbf{r}_0^2) + M^3 + 2\mathbf{r}_0(M^2 + 2M\mathbf{r} - \mathbf{r}^2 - \mathbf{r}_0^2) \cos \theta \\ &\quad [(3\mathbf{r} - M)\mathbf{r}_0^2 + (\mathbf{r} - M)M^2] \cos^2 \theta - 2\mathbf{r}_0 M^2 \cos^3 \theta \} \end{aligned} \quad (174c)$$

We now compute the change in the black-hole area for the perturbed metric associated with Ψ_P . Because the perturbed solution is static, the horizon is again a Killing horizon. Because the identification of perturbed and unperturbed spacetimes has been chosen to keep t^α as the Killing vector of the perturbed spacetime, the horizon is outermost set of points where t^α is null, where $t^\alpha t_\alpha = g_{tt} = 0$. Then, using $h_{tt} = h_{22}$ (this is a consequence of the

gauge, $h_{\alpha\beta}l^\beta = 0$, and the definition of the tetrad), we obtain the perturbation \dot{r} in the horizon radius at fixed θ and ϕ by writing

$$\begin{aligned} 0 &= \frac{d}{d\zeta} g_{tt}[\zeta, r(\zeta)] = \frac{d}{d\zeta} \left[1 - \frac{2M}{r(\zeta)} \right] + h_{22}|_{r=2M} = \frac{2M}{r^2} \Big|_{r=2M} \dot{r} + h_{22}|_{r=2M} \\ &= \frac{1}{2M} \dot{r} + A[r_0 - M(1 + \cos \theta)], \end{aligned} \quad (175)$$

whence

$$\dot{r} = -2AM[r_0 - M(1 + \cos \theta)]. \quad (176)$$

The change in the area of the horizon has, in general, contributions from the change \dot{r} in its position and the change in the element of area, $\frac{d}{d\zeta} \sqrt{2g} d\theta d\phi = \sqrt{2g} (h_{33} + h_{44}) d\theta d\phi$. In the radiation gauge, however, $h_{33} + h_{44}$ vanishes, and the change in the horizon area is given by

$$\dot{A}_{horizon} = \frac{d}{d\zeta} \int r_{horizon}^2 d\Omega = -2 \int r_{horizon} \dot{r} d\Omega \quad (177)$$

$$= -32\pi AM^2(r_0 - M). \quad (178)$$

The local first law of black hole thermodynamics relates the change in the Komar mass, $M_K = -(8\pi)^{-1} \int_S \nabla^\alpha t^\beta dS_{\alpha\beta}$, of a Killing horizon to the change in its area: $\dot{M} = \frac{1}{32\pi M} \dot{A}_{horizon}$. By Eq. (178), the particular solution associated with Ψ_P changes the mass of the horizon by

$$\dot{M} = -AM(r_0 - M). \quad (179)$$

We can thus keep area and mass of the horizon constant by adding to $h_{\alpha\beta}[\Psi_P]$ the $l = 0$ metric perturbation (156) associated with a change

$$\dot{M} = AM(r_0 - M) \quad (180)$$

of the Schwarzschild mass.

Note that instead of computing the change in area, we could have directly computed the change in the horizon's Komar mass.

As mentioned earlier, in addition to the singular radiation-gauge forms of the mass perturbation associated with b_1 and b_2 , there is a radiation gauge in which the mass perturbation is smooth on the exterior Schwarzschild spacetime. Choosing a gauge vector ξ^α with components $\xi_2 = \frac{2M}{r} - \frac{\Delta}{r^2} \ln\left(\frac{r}{2M} - 1\right)$, $\xi_1 = 2\frac{r^2}{\Delta}\xi_2$, we obtain from Eq. (180) a perturbed metric $h_{\alpha\beta} \equiv h_{\alpha\beta}^S - \mathcal{L}_\xi g_{\alpha\beta}$ whose only nonzero component is $h_{22} = -\frac{2}{r}$.

F. Singularity

For the static charge of the last section and for a static mass on a flat background (treated in the next section) the perturbed electromagnetic and gravitational fields are singular only at the position of the particle, although any choice of radiation gauge yields components of A_α and $h_{\alpha\beta}$, respectively, that are singular on (at least) a ray whose endpoint is the particle. For a static mass in a Schwarzschild background, we have seen that the perturbed gauge-invariant components of the Weyl tensor are also nonsingular except at the particle. This is surprising, because the metric perturbation must have an additional singularity that can be regarded as a string or strut supporting the particle. An initially static mass with no external support will fall inward: If the linearized Einstein equations are satisfied for a source of the form $\dot{T}^{\alpha\beta} = \dot{\rho}u^\alpha u^\beta$, with u^α along the timelike Killing vector that Lie derives $\delta\rho$, then the perturbed Bianchi identities imply

$$0 = \nabla_\beta \dot{T}^{\alpha\beta} = \dot{\rho}u^\beta \nabla_\beta u^\alpha. \quad (181)$$

It follows that the particle moves along a geodesic of the background geometry, $u^\beta \nabla_\beta u^\alpha = 0$, contradicting the assumption that the particle is static. We will find that the perturbed geometry has a conical singularity, and it can be chosen to run along a radial line from the particle to infinity. There is also, associated with any radiation gauge, a singularity of the perturbed metric that, like that of the vector potential in the last section, lies along a ray with endpoint at the particle and that can be removed by a gauge transformation.

We now show that one can choose the parameters a_1, \dots, d_2 to make the components of the metric finite either on a radial ray above the particle ($r > r_0$) or below the particle ($r < r_0$), but not both. If one departs from a radiation gauge, one can make the metric finite above and below the particle, but one cannot avoid a conical singularity.

Although the most efficient way to obtain necessary and sufficient conditions for smoothness of the perturbed metric is again to use the Hertz potential, in the case at hand the perturbed metric will not be smooth. We can at best find a gauge in which its components are finite, and, for clarity of presentation, we will use the direct expansion of the components of $h_{\alpha\beta}$.

The components of the perturbed metric near $\theta = 0$, are given for $r < r_0$ by

$$h_{33} = A \frac{4}{r\theta^2} [-2M^2a_1 + b_2 + c_2 + r_0^2 + (b_1 + c_1 - 3Md_1 - r_0)r] \\ - A \frac{2}{3r} \{-2M^2a_1 + b_2 + c_2 + r_0^2 \\ + [3M(-a_1 + 3d_1 + 1) + b_1 + c_1 - 3Md_1 - r_0]r\} + O(\theta^2), \quad (182a)$$

$$h_{23} = A \frac{\sqrt{2}(r - 2M)}{r^2\theta} (2M^2a_1 - b_2 - c_2 - r_0^2) + O(\theta^0); \quad (182b)$$

and for $r > r_0$ by

$$h_{33} = A \frac{4}{r\theta^2} [-2M^2a_1 + b_2 + c_2 - r_0^2 + (b_1 + c_1 - 3Md_1 + r_0)r] \\ - A \frac{2}{3r} \{-2M^2a_1 + b_2 + c_2 - r_0^2 \\ + [3M(-a_1 + 3d_1 - 1) + b_1 + c_1 - 3Md_1 + r_0]r\} + O(\theta^2), \quad (183a)$$

$$h_{23} = A \frac{\sqrt{2}(r - 2M)}{r^2\theta} (2M^2a_1 - b_2 - c_2 + r_0^2) + O(\theta^0); \quad (183b)$$

Then $h_{\alpha\beta}$ is smooth at $\theta = 0$ if and only if the coefficients of θ^{-2} and θ^0 in h_{33} and the coefficient of θ^{-1} in h_{23} vanish. The corresponding conditions on the parameters are

$$2M^2a_1 - b_2 - c_2 = r_0^2, \quad b_1 + c_1 - 3Md_1 = r_0, \quad a_1 - 3d_1 = 1, \quad r < r_0; \quad (184a)$$

$$2M^2a_1 - b_2 - c_2 = -r_0^2, \quad b_1 + c_1 - 3Md_1 = -r_0, \quad a_1 - 3d_1 = -1, \quad r > r_0. \quad (184b)$$

Because the conditions for $r < r_0$ are not consistent with those for $r > r_0$, one cannot find an single radiation gauge smooth at $\theta = 0$ both above and below the particle.

Near $\theta = \pi$, the perturbed metric has components

$$h_{33} = A \frac{4}{r(\pi - \theta)^2} [2M^2a_1 + b_2 - c_2 - (r_0 - 2M)^2 + (b_1 - c_1 - 3Md_1 - r_0 + 2M)r] \\ - A \frac{2}{3r} \{2M^2a_1 + b_2 - c_2 - (r_0 - 2M)^2 + [b_1 - c_1 - 3Md_1 - r_0 + 2M + 3M(a_1 + 3d_1 - 1)]r\} \\ + O[(\pi - \theta)^2], \quad (185a)$$

$$h_{23} = \frac{\sqrt{2}(r - 2M)A}{r^2(\pi - \theta)} [2M^2a_1 + b_2 - c_2 - (r_0 - 2M)^2] + O[(\pi - \theta)]. \quad (185b)$$

Then $h_{\alpha\beta}$ is smooth at $\theta = \pi$ if and only if the coefficients of $(\pi - \theta)^{-2}$ and $(\pi - \theta)^0$ in h_{33} and the coefficient of $(\pi - \theta)^{-1}$ in h_{23} vanish. The corresponding conditions on the parameters are

$$2M^2a_1 + b_2 - c_2 = (r_0 - 2M)^2, \quad b_1 - c_1 - 3Md_1 = (r_0 - 2M), \\ a_1 + 3d_1 = 1. \quad (186)$$

Requiring the perturbed metric to be either
(I) smooth everywhere except $\theta = 0$, $r > r_0$, or
(II) smooth everywhere except $\theta = 0$, $r < r_0$

uniquely determines the subset of parameters $\{a_1, b_1, b_2, c_1, c_2, d_1\}$. For the two cases, these parameters – the unique solutions to Eqs. (186) and to either Eqs. (184a) or (184b) – are listed in Table IV. The parameter a_2 does not affect the singular structure of $h_{\alpha\beta}$ and is

TABLE IV: Singularity location for two choices of parameters

Parameter	(I) Singularity at $\theta = 0, r \geq r_0$	(II) Singularity at $\theta = 0, r \leq r_0$
a_1	1	0
a_2	0	0
b_1	$(r_0 - M)$	0
b_2	$-2M(r_0 - M)$	$r_0^2 - 2Mr_0 + 2M^2$
c_1	M	$-r_0 + M$
c_2	$-r_0(r_0 - 2M)$	$2M(r_0 - M)$
d_1	0	$\frac{1}{3}$

arbitrary; for definiteness it is set to zero in the table. The trivial parameter d_2 does not alter the metric and is not listed.

The perturbed metric $h_{\alpha\beta}^{(I)}$ corresponding to the parameters of the first column is, in particular, smooth for $r < r_0$. The nonzero values of the parameters b_1 and b_2 correspond to a perturbation of the Schwarzschild mass given by

$$\dot{M} = 3AMb_1 + Ab_2 = AM(r_0 - M). \quad (187)$$

This is exactly the change in mass (180) needed to cancel the change in black hole area and mass of the particular solution. Smoothness of an IRG (or ORG) gauge in the region $2M < r < r_0$ between the horizon and the particle thus requires that the perturbation involve no change in the black hole mass. Just as the vector potential in a radiation gauge is smooth on a sphere only if the sphere encloses no perturbed charge, our result suggests that the perturbed metric can be smooth on a sphere only if the sphere encloses no perturbed mass.

We thus expect that for the perturbed metric $h_{\alpha\beta}^{(II)}$, smooth for $r > r_0$ (and asymptotically flat), the asymptotic mass must vanish, and direct calculation verifies this. For a flat background, we will see in the next section that one can obtain from $h_{\alpha\beta}^{(II)}$ a perturbed metric that agrees up to gauge with $h_{\alpha\beta}^{(I)}$ by adding an $l = 0$ part, a perturbed Schwarzschild solution centered at the origin that restores the total asymptotic mass. For a Schwarzschild background, however, there is a second physical difference between solutions (I) and (II): The difference in the values of a_1 means that $h_{\alpha\beta}^{(I)}$ and $h_{\alpha\beta}^{(II)}$ differ by a perturbed C-metric. The solution that agrees up to gauge with $h_{\alpha\beta}^{(I)}$ has the form $h_{\alpha\beta}^{(II)} + h_{\alpha\beta}^{\text{Schwarzschild}} + h_{\alpha\beta}^{C\text{-metric}}$. The C-metric term has a conical singularity on the axis; adding it in the gauge of [14], and adding the Schwarzschild perturbation in a Schwarzschild gauge yields a perturbed metric that is finite but not smooth on the axis.

V. FLAT SPACE

When the mass M of the background Schwarzschild geometry is set to zero, the perturbation is just the gravitational field of a point mass linearized about a flat background: the linearized Schwarzschild solution. Like the electromagnetic field of a point charge in a Schwarzschild background, the perturbed geometry is now singular only at the position of the particle, and any other singularity in the tensor $h_{\alpha\beta}$ is an artifact of the choice of gauge.

The origin of coordinates inherited from the $M \neq 0$ case is displaced from the particle by a distance r_0 ; and the inherited tetrad is aligned with inward and outward radial null geodesics associated with the spatial origin, not with the particle. With respect to a tetrad associated with the null geodesics that start or end *at the particle*, the perturbation is algebraically special, having as the only nonzero component of its perturbed Weyl tensor $\psi_2 = -\frac{\mathfrak{m}}{R^3}$, with R the distance to the particle. With respect to our null tetrad, however, the perturbation

is not algebraically special: Its components are

$$\psi_0 = -3 \frac{\mathfrak{m}}{R^5} r_0^2 \sin^2 \theta, \quad (188a)$$

$$\psi_1 = \frac{3}{\sqrt{2}} \frac{\mathfrak{m}}{R^5} r_0 (r - r_0 \cos \theta) \sin \theta \quad (188b)$$

$$\psi_2 = -\frac{\mathfrak{m}}{R^5} \left(R^2 - \frac{3}{2} r_0^2 \sin^2 \theta \right), \quad (188c)$$

$$\psi_3 = -\frac{3}{2\sqrt{2}} \frac{\mathfrak{m}}{R^5} r_0 (r - r_0 \cos \theta) \sin \theta \quad (188d)$$

$$\psi_4 = -\frac{3}{4} \frac{\mathfrak{m}}{R^5} r_0^2 \sin^2 \theta. \quad (188e)$$

where

$$R = \sqrt{r^2 + r_0^2 - 2rr_0 \cos \theta}. \quad (189)$$

(This coordinate expression for the distance R to the particle is the $M = 0$ limit of the length R of the last section.) In contrast to perturbations of Kerr and Schwarzschild, where only ψ_0 and ψ_4 are gauge invariant, the full Weyl tensor is gauge invariant, because the Weyl tensor of the flat background vanishes. The corresponding components along the null tetrad $l^\alpha, n^\alpha, m'^\alpha$ differ from these only by a phase and are manifestly smooth except at $R = 0$, the position of the particle, and at $r = 0$, where the tetrad is singular.

$$\psi'_0 = -3 \frac{\mathfrak{m}}{R^5} r_0^2 \sin^2 \theta e^{2i\phi} = -3 \frac{\mathfrak{m}}{R^5} \frac{r_0^2}{r^2} (x + iy)^2, \quad (190a)$$

$$\psi'_1 = \frac{3}{\sqrt{2}} \frac{\mathfrak{m}}{R^5} r_0 (r - r_0 \frac{z}{r}) \sin \theta e^{i\phi} = \frac{3}{\sqrt{2}} \frac{\mathfrak{m}}{R^5} \frac{r_0}{r} (r - \frac{r_0}{r} z) (x + iy) \quad (190b)$$

$$\psi'_2 = -\frac{\mathfrak{m}}{R^5} \left(R^2 - \frac{3}{2} r_0^2 \sin^2 \theta \right) = -\frac{\mathfrak{m}}{R^5} \left[R^2 - \frac{3}{2} \frac{r_0^2}{r^2} (x^2 + y^2) \right], \quad (190c)$$

$$\psi'_3 = -\frac{3}{2\sqrt{2}} \frac{\mathfrak{m}}{R^5} r_0 (r - r_0 \cos \theta) \sin \theta e^{-i\phi} = -\frac{3}{2\sqrt{2}} \frac{\mathfrak{m}}{R^5} \frac{r_0}{r} (r - \frac{r_0}{r} z) (x - iy) \quad (190d)$$

$$\psi'_4 = -\frac{3}{4} \frac{\mathfrak{m}}{R^5} r_0^2 \sin^2 \theta = -\frac{3}{4} \frac{\mathfrak{m}}{R^5} \frac{r_0^2}{r^2} (x - iy)^2. \quad (190e)$$

The components ψ'_i are smooth at $\theta = \pi$, although the tetrad is not; this is due to the fact that the tetrad $l^\alpha, n^\alpha, e^{-i\phi} m^\alpha$ is smooth at $\theta = \pi$, and the components of the Weyl tensor associated with that tetrad are simply $\bar{\psi}'_i$.

The perturbed mass is given by

$$\begin{aligned} \lim_{r \rightarrow \infty} (-r^3 \psi_2) &= \lim_{r \rightarrow \infty} r^3 \frac{\mathfrak{m}}{R^5} \left(R^2 - \frac{3}{2} r_0^2 \sin^2 \theta \right) \\ &= \mathfrak{m}. \end{aligned} \quad (191)$$

We will again find the perturbed metric $h_{\alpha\beta}$ in a radiation gauge, choosing, within the family of radiation gauges, one for which the metric components are finite except on the $\theta = 0$ axis for $r > r_0$ or for $r < r_0$. As was the case for the vector potential of the smooth electromagnetic field of a point particle, one cannot globally choose parameters a_1, \dots, d_2 for which the metric is smooth and regular everywhere outside the particle.

As in the previous sections, if the Hertz potential Ψ' associated with the smooth tetrad $l^\alpha, n^\alpha, m'^\alpha$ is smooth at $\theta = 0$, then the perturbed metric associated with this tetrad is smooth at $\theta = 0$.

From Eqs. (140) and (141), a particular solution Ψ_P associated with the Kinnersley tetrad has the form

$$\Psi_P = -\frac{2}{3}A\frac{R^3}{\sin^2\theta}, \quad (192)$$

and the general homogeneous solution has the form

$$\begin{aligned} \Psi_H = A\frac{1}{3} [& (a_1 \cos^3\theta - 3a_1 \cos\theta + 6d_1)r^3 + 3(b_1 \cos^2\theta + b_1 + 2c_1 \cos\theta)r^2 \\ & + (a_2 \cos^3\theta - 3a_2 \cos\theta + 6d_2) + 3(b_2 \cos^2\theta + b_2 + 2c_2 \cos\theta)r]. \end{aligned} \quad (193)$$

The Hertz potentials associated with tetrads smooth at $\theta = 0$ and $\theta = \pi$ are then $\Psi e^{\pm 2i\phi} = (\Psi_P + \Psi_H)e^{\pm 2i\phi}$.

We will show that one can make the corresponding metric $h_{\alpha\beta}$ smooth at $\theta = \pi$, by a choice of parameters that correspond to gauge transformations. One can simultaneously choose parameters to make $h_{\alpha\beta}$ smooth at $\theta = 0$ for either $r \geq 0$ or for $r \leq 0$. But no consistent choice of parameters allows the perturbed metric to be smooth both above and below the particle on the ray through the particle. Moreover, smoothness at $\theta = 0$ above the particle requires a choice of parameters that alters the asymptotic mass; to recover for $r > r_0$ the metric of a static particle, one must add to the IRG solution an $l = 0$ part, a linearized Schwarzschild solution centered at the origin.

We begin by describing changes in the perturbed metrics associated with the parameters a_1, \dots, d_2 that arise for a flat background. For Kerr geometries with nonzero mass, Wald [32] shows that any perturbation with ψ_0 or $\psi_4 = 0$ is a type D perturbation or is pure gauge. Because the proof relies on the fact that a gauge transformation can make ψ_1 and ψ_3 vanish, it fails for a flat background. There are, in fact, vacuum perturbations of flat space for which the gauge invariant components ψ_1, ψ_2 and ψ_3 have nonzero values, although $\psi_0 = \psi_4 = 0$. In particular, we will see that the perturbation associated with a_2 is no longer

pure gauge but is a perturbation of this kind. (This can happen because the gauge vector (147) has no $M = 0$ limit).

Conversely, we will see that the perturbations associated with a_1 and b_2 are now pure gauge, although they are C-metric and mass perturbations when $M \neq 0$. The components of the perturbed metric and Weyl tensor associated with each of the parameters can be read off from the corresponding expressions in the last section by setting M to zero. Because each parameter corresponds to a vacuum perturbation and the background Riemann tensor vanishes, the perturbation is pure gauge if and only if the components of the perturbed Weyl tensor vanish. That is, a perturbation of flat space is pure gauge iff the perturbed Riemann tensor $\dot{R}^\alpha_{\beta\gamma\delta}$ vanishes; and, for a perturbation of flat space, we have

$$\dot{R}^\alpha_{\beta\gamma\delta} = 0 \Leftrightarrow \dot{R}_{\alpha\beta\gamma\delta} = 0 \Leftrightarrow \dot{R}_{\alpha\beta} = 0 \text{ and } \dot{C}_{\alpha\beta\gamma\delta} = 0.$$

Eqs. (144) imply that the perturbed Weyl tensor associated with a_1 vanishes, and thus that a_1 corresponds to a gauge perturbation, $h_{\alpha\beta} = \mathcal{L}_\xi g_{\alpha\beta}$.

Writing $\tilde{A} := A|_{M=0} = \frac{\mathfrak{m}}{r_0^2}$, we obtain

$$h_{22} = -a_1 \tilde{A} 2r \cos \theta \quad (194a)$$

$$h_{23} = -a_1 \tilde{A} \frac{1}{\sqrt{2}} r \sin \theta \quad (194b)$$

$$h_{33} = 0, \quad (194c)$$

$$\xi_1 = a_1 \tilde{A} \frac{1}{2} (t-r)^2 \cos \theta \quad (195a)$$

$$\xi_2 = -a_1 \tilde{A} \frac{1}{4} (t-r)(t+3r) \cos \theta \quad (195b)$$

$$\xi_3 = -a_1 \tilde{A} \frac{1}{2\sqrt{2}} (t^2 - r^2) \sin \theta. \quad (195c)$$

Because the components (146) of the perturbed Weyl tensor associated with a_2 are nonzero, the perturbation associated with a_2 is no longer pure gauge.

The perturbation associated with b_1 was a change of mass proportional to M ; because Eqs. (155) imply $\psi_i = 0$, it is pure gauge when $M = 0$:

$$h_{22} = -2b_1 \tilde{A} \quad (196a)$$

$$h_{23} = 0 \quad (196b)$$

$$h_{33} = b_1 \tilde{A} \frac{2(1 + \cos^2 \theta)}{\sin^2 \theta} \quad (196c)$$

$$\xi_1 = b_1 \tilde{A} (r - t) \quad (197a)$$

$$\xi_2 = b_1 \tilde{A} \frac{1}{2} (3r - t) \quad (197b)$$

$$\xi_3 = -b_1 \tilde{A} \sqrt{2} r \cot \theta. \quad (197c)$$

The metric associated with b_2 remains a mass perturbation, a linearized Schwarzschild solution, centered at the origin with mass $-\mathbf{m}b_2/r_0^2$. Finally, the perturbations associated with c_1 and c_2 are again pure gauge; and the perturbations associated with *both* d_1 and d_2 now vanish, $h_{\alpha\beta} = 0$. To summarize: In flat space the perturbations associated with a_1, b_1, c_1, c_2 are pure gauge, and those associated with d_1 and d_2 vanish.

Then one can find a smooth perturbed metric of a static particle in flat space from an IRG Hertz potential only if one can choose the six parameters $a_1, b_1, c_1, c_2, d_1, d_2$ to make the perturbed metric smooth. Now $h_{\alpha\beta}$ is smooth on the axis if and only if its components in a smooth basis are smooth. Because $h_{2'3'}$ and $h_{3'3'}$ have angular dependence $e^{im\phi}$ and $e^{2im\phi}$, they are smooth at $\theta = 0$ only if $h_{22} = O(\theta^0), h_{23'} = O(\theta), h_{3'3'} = O(\theta^2)$; and this is true if and only if

$$h_{22} = O(\theta^0), \quad h_{23} = O(\theta), \quad h_{33} = O(\theta^2). \quad (198)$$

Similarly, the metric can be smooth at $\theta = \pi$ only if

$$h_{22} = O((\theta - \pi)^0), \quad h_{23} = O((\theta - \pi)), \quad h_{33} = O((\theta - \pi)^2). \quad (199)$$

These requirements impose more algebraic conditions on the parameters than can be simultaneously satisfied. One can, however, choose parameters for which the metric is smooth everywhere except on a ray that does not intersect the particle (and at the particle itself): That is one can find a perturbed metric from a Hertz potential Ψ' that is smooth everywhere except $\theta = \pi$.

As we saw with a Schwarzschild background, one can also choose parameters that make the perturbation regular in a radiation gauge either inside or outside the particle. Inside the particle, the choice is simply a gauge transformation. Any choice of parameters that make the perturbation regular outside the particle require zero net mass for $r \geq r_0$. One can obtain a regular solution in a radiation gauge for the $l \neq 0$ part of the perturbation; to complete the solution one must add an $l = 0$ part – corresponding to a mass at the origin – in a different gauge.

One can find the allowed parameters either by directly expanding the metric for arbitrary parameter values near $\theta = 0$ and $\theta = \pi$ or, more simply, by examining the Hertz potential near $\theta = 0$ and $\theta = \pi$. Now Ψ' is smooth near $\theta = 0$ ($\theta = \pi$) if and only if $\Psi = O(\theta^2)$ ($\Psi = O((\theta - \pi)^2)$). From Eqs. (192,193), the parts Ψ_P and Ψ_H of the Hertz potential have near $\theta = 0$ the forms

$$\Psi_P = -\tilde{A}\frac{2}{3}|r - r_0| \left[(r - r_0)^2\theta^{-2} + \frac{1}{3}(r - r_0)^2 + \frac{3}{2}r_0r \right] + O(\theta^2), \quad (200a)$$

$$\begin{aligned} \Psi_H = \tilde{A}\frac{2}{3} \left\{ [(-a_1 + 3d_1)r^3 + 3(b_1 + c_1)r^2 + 3(b_2 + c_2)r - a_2 + 3d_2] \theta^{-2} \right. \\ \left. + \left[\frac{1}{3}(-a_1 + 3d_1)r^3 - \frac{1}{2}(b_1 + c_1)r^2 - \frac{1}{2}(b_2 + c_2)r + \frac{1}{3}(-a_2 + 3d_2) \right] \right\} + O(\theta^2). \end{aligned} \quad (200b)$$

Then, for $\Psi = \Psi_P + \Psi_H$, we have

$$\begin{aligned} \Psi = \tilde{A}\frac{2}{3} \left\{ [(-1 - a_1 + 3d_1)r^3 + 3(r_0 + b_1 + c_1)r^2 + 3(-r_0^2 + b_2 + c_2)r + r_0^3 - a_2 + 3d_2] \theta^{-2} \right. \\ \left. + \left[\frac{1}{3}(-1 - a_1 + 3d_1)r^3 - \frac{1}{2}(r_0 + b_1 + c_1)r^2 - \frac{1}{2}(-r_0^2 + b_2 + c_2)r + \frac{1}{3}(r_0^3 - a_2 + 3d_2) \right] \right\} \\ + O(\theta^2), \quad r > r_0, \end{aligned} \quad (201a)$$

$$\begin{aligned} \Psi = \tilde{A}\frac{2}{3} \left\{ [(1 - a_1 + 3d_1)r^3 + 3(-r_0 + b_1 + c_1)r^2 + 3(r_0^2 + b_2 + c_2)r - r_0^3 - a_2 + 3d_2] \theta^{-2} \right. \\ \left. + \left[\frac{1}{3}(1 - a_1 + 3d_1)r^3 - \frac{1}{2}(-r_0 + b_1 + c_1)r^2 - \frac{1}{2}(r_0^2 + b_2 + c_2)r + \frac{1}{3}(-r_0^3 - a_2 + 3d_2) \right] \right\} \\ + O(\theta^2), \quad r < r_0. \end{aligned} \quad (201b)$$

Similarly, for θ near π , we have

$$\begin{aligned} \Psi = \tilde{A}\frac{2}{3} \left\{ [(-1 + a_1 + 3d_1)r^3 + 3(-r_0 + b_1 - c_1)r^2 + 3(-r_0^2 + b_2 - c_2)r - r_0^3 + a_2 + 3d_2] (\theta - \pi)^{-2} \right. \\ \left. + \left[\frac{1}{3}(-1 + a_1 + 3d_1)r^3 - \frac{1}{2}(-r_0 + b_1 - c_1)r^2 - \frac{1}{2}(-r_0^2 + b_2 - c_2)r + \frac{1}{3}(-r_0^3 + a_2 + 3d_2) \right] \right\} \\ + O[(\theta - \pi)^2]. \end{aligned} \quad (202)$$

We thus obtain as necessary and sufficient conditions for smoothness of Ψ at $\theta = 0$,

$$a_1 - 3d_1 = -1, \quad a_2 - 3d_2 = r_0^3, \quad b_1 + c_1 = -r_0, \quad b_2 + c_2 = r_0^2, \quad r > r_0; \quad (203a)$$

$$a_1 - 3d_1 = 1, \quad a_2 - 3d_2 = -r_0^3, \quad b_1 + c_1 = r_0, \quad b_2 + c_2 = -r_0^2, \quad r < r_0; \quad (203b)$$

and at $\theta = \pi$,

$$a_1 + 3d_1 = 1, \quad a_2 + 3d_2 = r_0^3, \quad b_1 - c_1 = r_0, \quad b_2 - c_2 = r_0^2. \quad (204)$$

Because the perturbations corresponding to d_1 and d_2 vanish, one can satisfy the first two conditions in each line without changing $h_{\alpha\beta}$. The necessary and sufficient conditions for smoothness of $h_{\alpha\beta}$ at $\theta = 0$ (as can be verified directly from its components) are then

$$b_1 + c_1 = -r_0, \quad b_2 + c_2 = r_0^2, \quad r > r_0; \quad (205a)$$

$$b_1 + c_1 = r_0, \quad b_2 + c_2 = -r_0^2, \quad r < r_0; \quad (205b)$$

and at $\theta = \pi$,

$$b_1 - c_1 = r_0, \quad b_2 - c_2 = r_0^2. \quad (206)$$

As in the last section, Eqs. (205) immediately imply that one cannot find a radiation gauge (a gauge that can be everywhere locally obtained from an IRG Hertz potential) for which $h_{\alpha\beta}$ is simultaneously smooth at $\theta = 0$ outside and inside the particle. One can choose parameters for which $h_{\alpha\beta}$ smooth everywhere except

- (I) along the part of the $\theta = 0$ ray below the particle (with $r \leq r_0$), or
- (II) along the part of the $\theta = 0$ ray above the particle (with $r \geq r_0$). For the parameter choice that makes $h_{\alpha\beta}$ smooth outside $r = r_0$, however, b_2 has the nonzero value $-r_0^2$, changing the asymptotic mass to zero by subtracting a mass \mathbf{m} Schwarzschild solution centered at the origin. As in the Schwarzschild perturbation of the last section, one can then obtain a smooth metric outside $r = r_0$ with asymptotic mass \mathbf{m} by adding back an $l = 0$ perturbation in a smooth gauge – adding, for example the linearized Schwarzschild solution of mass \mathbf{m} centered at the origin.

TABLE V: Singularity location for two choices of parameters

Parameter	(I) Singularity at $\theta = 0; r \geq r_0$	(II) Singularity at $\theta = 0; r \leq r_0$
a_1	0	0
a_2	0	0
b_1	r_0	0
b_2	0	r_0^2
c_1	0	$-r_0$
c_2	$-r_0^2$	0

The perturbed metric smooth everywhere except $\theta = 0$, $r \geq r_0$ is given by

$$h_{22}^{(I)} = -2\mathbf{m} \left(\frac{1}{r_0} - \frac{1}{R} \right) \quad (207a)$$

$$h_{23}^{(I)} = -\sqrt{2} \mathbf{m} \frac{1}{\sin \theta} \left(\frac{r_0}{rR} - \frac{1}{r} - \frac{1}{R} \cos \theta \right) \quad (207b)$$

$$h_{33}^{(I)} = -2\mathbf{m} \frac{1}{\sin^2 \theta} \left[\frac{1}{R} - \frac{1}{r_0} + 2 \left(\frac{1}{r} - \frac{r_0}{rR} - \frac{r}{r_0 R} \right) \cos \theta + \left(\frac{3}{R} - \frac{1}{r_0} \right) \cos^2 \theta \right]. \quad (207c)$$

The perturbed metric smooth everywhere except $\theta = 0$, $r \leq r_0$ is given by

$$h_{22}^{(II)} = -2\mathbf{m} \left(\frac{1}{r} - \frac{1}{R} \right) \quad (208a)$$

$$h_{23}^{(II)} = -\sqrt{2} \mathbf{m} \frac{1}{\sin \theta} \left[\frac{r_0}{rR} + \left(\frac{1}{r} - \frac{1}{R} \right) \cos \theta \right] \quad (208b)$$

$$h_{33}^{(II)} = -2\mathbf{m} \frac{1}{\sin^2 \theta} \left[\frac{1}{R} - \frac{1}{r} + 2 \left(\frac{1}{r_0} - \frac{r}{r_0 R} - \frac{r_0}{rR} \right) \cos \theta + \left(\frac{3}{R} - \frac{1}{r} \right) \cos^2 \theta \right]. \quad (208c)$$

The perturbed metrics $h_{\alpha\beta}^{(I)}$ and $h_{\alpha\beta}^{(II)} + h_{\alpha\beta}^{\text{Schwarzschild}}(\mathbf{m})$ must agree, up to a gauge transformation, with the perturbation one acquires from the Schwarzschild metric by translating the origin and linearizing about flat space. This is the case, and the explicit gauge vector ξ^α for the metric $h_{\alpha\beta}^{(I)}$ has components

$$\xi_1 = -\mathbf{m} \left(\frac{t-r}{r_0} - \frac{r-r_0 \cos \theta}{R} + 2 \ln \frac{r-r_0 \cos \theta + R}{\sin^2 \frac{\theta}{2}} \right) \quad (209a)$$

$$\xi_2 = -\frac{\mathbf{m}}{2} \left(\frac{t-3r}{r_0} + \frac{r-r_0 \cos \theta}{R} + 2 \ln \frac{r-r_0 \cos \theta + R}{\sin^2 \frac{\theta}{2}} \right) \quad (209b)$$

$$\xi_3 = -\frac{\mathbf{m}}{2\sqrt{2} \sin \theta} \left\{ \frac{4r}{r_0} \cos \theta + \frac{2[R^2 + (r-r_0 \cos \theta)^2]}{r_0 R} - 4 \right\} \quad (209c)$$

$$\xi_\phi = 0. \quad (209d)$$

The resultant metric is given by

$$h_{11} = \frac{2\mathfrak{m}}{R^3}(2R^2 - r_0^2 \sin^2 \theta) \quad (210a)$$

$$h_{22} = \frac{\mathfrak{m}}{2R^3}(2R^2 - r_0^2 \sin^2 \theta) \quad (210b)$$

$$h_{33} = \frac{\mathfrak{m}r_0^2 \sin^2 \theta}{R^3} \quad (210c)$$

$$h_{12} = \frac{\mathfrak{m}r_0^2 \sin^2 \theta}{R^3} \quad (210d)$$

$$h_{13} = \frac{\sqrt{2}\mathfrak{m}r_0(r - r_0 \cos \theta) \sin \theta}{R^3} \quad (210e)$$

$$h_{23} = \frac{\mathfrak{m}r_0(r - r_0 \cos \theta) \sin \theta}{\sqrt{2}R^3} \quad (210f)$$

$$h_{34} = \frac{\mathfrak{m}r_0^2 \sin^2 \theta}{R^3} \quad (210g)$$

VI. FINDING THE SELF-FORCE IN A RADIATION GAUGE.

An algorithm for finding the self force on a mass moving in a background spacetime, the MiSaTaQuWa method, was given several years ago [20, 31]. In this method, as noted by Quinn and Wald and by Detweiler and Whiting [7], a particle follows a geodesic of a renormalized metric, h^{ren} , given by

$$h^{\text{ren}} = h^{\text{ret}} - h^{\text{sing}}, \quad (211)$$

where h^{ret} is the retarded field of the particle in a Lorenz gauge and h^{sing} is its locally defined singular part, chosen to cancel the singular part of h^{ret} and to give no contribution to the self-force. In the Quinn-Wald characterization of the method h^{sing} is obtained in a neighborhood of a point P of the particle's trajectory from the field of a particle moving in flat space by using the exponential map to relate the linearized field in flat space to a field h^{sing} in the neighborhood of P . One demands that the image under the exponential map of the flat-space particle's position, 4-velocity, and acceleration agree at P with those of the original particle.

Because it involves a Lorenz gauge, the method cannot take advantage of the decoupled, separable wave equation – the Teukolsky equation – that governs black hole perturbations. One could imagine constructing the singular part of the gauge-invariant perturbed Weyl

tensor component Ψ_0 (or Ψ_4) from h^{sing} and then obtaining the renormalized metric in a radiation gauge from Ψ_0^{ren} (or Ψ_4^{ren}). The radiation-gauge prescription, however, requires the perturbed Weyl scalar to be a solution to the linearized Einstein equation. In the forms in which it is given by Mino et al.[20] and by Quinn and Wald[31], h^{sing} is not a solution to the Einstein equations linearized about the background Kerr spacetime; and it is not obvious that one can obtain a prescription for a renormalized Weyl scalar Ψ_0^{ren} (or Ψ_4^{ren}) that satisfies the Teukolsky equation.

A more recent version of the MiSaTaQuWa method, due to Detweiler and Whiting [7], overcomes this difficulty. In their prescription, the singular part of the field h^{sing} is redefined in a way that makes it a solution to the linearized Einstein equations, defined in a normal neighborhood of the particle. As a result, one can compute a gauge-invariant renormalized Weyl scalar in the following way:

The perturbed Riemann tensor is given in terms of $h_{\alpha\beta}$ by

$$\dot{R}_{\alpha\beta\gamma\delta} = \frac{1}{2}(\nabla_\beta \nabla_\gamma h_{\alpha\delta} + \nabla_\alpha \nabla_\delta h_{\beta\gamma} - \nabla_\alpha \nabla_\gamma h_{\beta\delta} - \nabla_\beta \nabla_\delta h_{\alpha\gamma}) - R_{\alpha\beta[\gamma}{}^\epsilon h_{\delta]\epsilon}. \quad (212)$$

From Eq. (7), ψ_0 has the form,

$$\psi_0 = -l^\alpha m^\beta l^\gamma m^\delta \dot{C}_{\alpha\beta\gamma\delta} = -l^\alpha m^\beta l^\gamma m^\delta \dot{R}_{\alpha\beta\gamma\delta} = \mathcal{O}^{\alpha\beta} h_{\alpha\beta}, \quad (213)$$

where $\mathcal{O}^{\alpha\beta}$ is the operator

$$\mathcal{O}^{\alpha\beta} = \frac{1}{2}(l^\alpha l^\beta m^\gamma m^\delta + m^\alpha m^\beta l^\gamma l^\delta - l^\alpha m^\beta l^\gamma m^\delta - l^\alpha m^\beta m^\gamma l^\delta) \nabla_\gamma \nabla_\delta. \quad (214)$$

In particular, from the singular part of the perturbed metric, $h_{\alpha\beta}^{\text{sing}}$, one can define the singular part ψ_0^{sing} of the perturbed Weyl tensor by

$$\psi_0^{\text{sing}} := \mathcal{O}^{\alpha\beta} h_{\alpha\beta}^{\text{sing}}. \quad (215)$$

Then ψ_0^{ren} is given by

$$\psi_0^{\text{ren}} = \psi_0^{\text{ret}} - \psi_0^{\text{sing}}, \quad (216)$$

with

$$\psi_0^{\text{sing}} = \mathcal{O}^{\alpha\beta} h_{\alpha\beta}^{\text{sing}} \quad (217)$$

Because h^{ret} and h^{sing} have the same source, so do ψ^{ret} and ψ^{sing} . Their difference, ψ^{ren} is thus a *sourcefree* solution to the linearized Teukolsky equation. That implies that one can construct a nonsingular h^{ren} in a radiation gauge from Ψ^{ren} .

In a Lorenz gauge, we have

$$\bar{h}_{\alpha\beta}^{\text{sing}} = 4 \int G_{\alpha\beta\gamma'\delta'}^{\text{sing}}(x, x') T^{\gamma'\delta'}(x') d^4V', \quad (218)$$

where $\bar{h}^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}h$ and where the Green's function has the form

$$G_{\alpha\beta\gamma'\delta'}^{\text{sing}}(x, x') = \frac{1}{2}U_{\alpha\beta\gamma'\delta'}(x, x')\delta(\sigma) + \frac{1}{2}V_{\alpha\beta\gamma'\delta'}(x, x')\Theta(\sigma). \quad (219)$$

where U and V are nonsingular bitensors, whose form is given, for example, in [7] (see also [24]). The step function $\Theta(\sigma)$ has value 1 when $\sigma > 0$ – when x and x' are *spacelike* separated; and it vanishes for $\sigma < 0$. Thus $G_{\alpha\beta\gamma'\delta'}^{\text{sing}}(x, x')$ has peculiar causal behavior: It is nonzero only when x and x' are not timelike related.

A Green's function for Ψ_0^{sing} is then given by

$$G_{\gamma'\delta'}^{\text{sing}}(x, x') = \left(\mathcal{O}^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}\mathcal{O}_\epsilon^\epsilon \right) G_{\alpha\beta\gamma'\delta'}^{\text{sing}}(x, x'). \quad (220)$$

Having computed $G_{\gamma'\delta'}^{\text{sing}}(x, x')$, one can obtain ψ_0^{sing} from the expression

$$\psi_0^{\text{sing}}(x) = \int G_{\gamma'\delta'}(x, x') T^{\gamma'\delta'}(x') d^4V'. \quad (221)$$

The explicit forms of U and V have been computed in terms of the geodesic distance from the particle, to the order needed to find the self-force (although the order needed for the prescription given here may be different).

From ψ_0^{ren} one recovers $h_{\alpha\beta}^{\text{ren}}$, using the CCK equations (120) or (124) in a Kerr background to find the potential Ψ and using Eq. (111) to find h^{ren} in terms of Ψ . Eq. (120) can be solved by four radial integrations for each angular harmonic if one works in the frequency domain. The operator D , restricted to an m, ω subspace corresponding to ϕ and t dependence $e^{i(m\phi - \omega t)}$, has the form

$$D_{m\omega} = \partial_r + i \frac{[ma - (r^2 + a^2)\omega]}{\Delta}. \quad (222)$$

Then Ψ^{ren} is found from

$$\psi_{0, m\omega}^{\text{ren}} = D_{m\omega}^4 \Psi_{m\omega}^{\text{ren}}. \quad (223)$$

Because $h_{\alpha\beta}^{\text{ren}}$ is a solution to the linearized vacuum Einstein equations, it is determined by ψ_0 (or ψ_4) up to an algebraically special vacuum perturbation.

The method of finding the self-force in a radiation gauge may now be summarized as follows:

1. Compute ψ_0^{sing} either from the known form of h^{sing} as outlined above, or, with additional insight, directly from the Teukolsky equation.
2. Express ψ_0^{sing} in terms of spin-weighted spheroidal harmonics.
3. Compute ψ_0^{ret} from the Teukolsky equation.
4. Write $\psi_0^{\text{ren}} = \psi_0^{\text{ret}} - \psi_0^{\text{sing}}$, regularizing by a cutoff in angular harmonics, and finding the limit, to desired accuracy, as the cutoff harmonic is increased.
5. Find the potential Ψ either from ψ_4^{ren} , using Eq. (124), or from ψ_0^{ren} by 4 radial integrations of Eq. (223).
6. Find the perturbed metric $h_{\alpha\beta}$ in a radiation gauge by taking derivatives of Ψ . This leaves spin-weight 0 and 1 parts of the perturbation undetermined.
7. Obtain the spin-weight 0 and 1 parts of $h_{\alpha\beta}$ by fixing the area and angular velocity of the perturbed black hole and using jump conditions across the particle of the spin-weight 0 and 1 parts of the perturbed Einstein equation.⁴
8. Compute the self-force from the perturbed geodesic equation. For the perturbed metric $g_{\alpha\beta} + \zeta h_{\alpha\beta}$, with ∇ the covariant derivative of $g_{\alpha\beta}$ and u^α normalized by $g_{\alpha\beta}$, the geodesic equation has, to $O(\zeta)$, the form

$$u \cdot \nabla u^\alpha = -\zeta(g^{\alpha\delta} + u^\alpha u^\delta) \left(\nabla_\beta h_{\gamma\delta}^{\text{ren}} - \frac{1}{2} \nabla_\delta h_{\beta\gamma}^{\text{ren}} \right) \equiv f^\alpha. \quad (224)$$

With the self-force f^α computed, one must find the particle trajectory as a self-consistent solution to Eq. (224). The emitted radiation can be found from ψ_4^{ret} .

Because the perturbed Weyl tensor involves two derivatives of the perturbed metric, it is possible that the renormalization program outlined here will require a Hadamard expansion that is two orders higher in the separation between source and field point than the expansion used in a Lorenz gauge. We suspect, however, that the extra orders can be avoided by using Eq. (223) to find the Hertz potential. The argument is that each radial integration changes by one factor of R the highest power of $1/R$ and thereby reduces the order of the singularity.

⁴ Using the jump conditions of the spin-weight 0 and 1 parts of the Einstein equation was suggested to us by Bernard Whiting.

Calculations now underway in a Schwarzschild background will decide the issue and may show whether the method can feasibly be used to obtain generic orbits in a Kerr background.

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APPENDIX A: SMOOTHNESS IN TERMS OF A HERTZ POTENTIAL

In the CCK formalism one writes the vector potential A_α and the perturbed metric $h_{\alpha\beta}$ in terms of Hertz potentials Φ and Ψ . The components of A_α and $h_{\alpha\beta}$, given by Eqs. (61) and (118), have in each case the form $\mathcal{L}\Phi$ (or $\mathcal{L}\Psi$), where \mathcal{L} is a sum of covariant derivatives along the basis vectors and Christoffel symbols (spin coefficients) of the basis. Because the Christoffel symbols of a smooth basis are smooth, smoothness of the Hertz potential Φ^\pm associated with a smooth basis $\{\mathbf{e}_\mu^\pm\}$ is a sufficient condition for smoothness of A_α ; and smoothness of the Hertz potential Ψ^\pm associated with a smooth basis is a sufficient condition for smoothness of $h_{\alpha\beta}$.

We next relate this condition to a condition on the Hertz potential associated with the Kinnersley basis, by showing that the Hertz potentials Φ^\pm and Ψ^\pm associated with the smooth bases $\{\mathbf{e}_\mu^\pm\}$ differ only by a phase from the Hertz potentials Φ and Ψ . In addition, the metrics corresponding to the two sets of Hertz potentials are identical.

These results are immediate from the following proposition, proved here for a Schwarzschild geometry, but presumably correct for type D vacuum spacetimes.

Proposition. (i) Let Ψ satisfy the sourcefree Bardeen-Press (Teukolsky) equation for spin-weight $s = 2$, namely $\mathcal{T}\bar{\Psi} = 0$. Then $\Psi^\pm = \Psi e^{\mp 2i\phi}$ satisfies the same sourcefree equation, $\mathcal{T}\bar{\Psi}^\pm = 0$, and $h_{\alpha\beta}[\Psi^\pm] = h_{\alpha\beta}[\Psi]$.

(ii) Let Φ satisfy the sourcefree Teukolsky equation for spin-weight $s = 1$, $\mathcal{T}\bar{\Phi} = 0$. Then $\Phi^\pm = \Phi e^{\mp i\phi}$ satisfies the same sourcefree equation, $\mathcal{T}\bar{\Phi}^\pm = 0$, and $A_\alpha[\Phi^\pm] = A_\alpha[\Phi]$.

Proof. The proof essentially follows from the fact that expressions arising in the NP and CCK formalism have definite spin-weight. A Lorentz tranformation of the basis that changes only the phase of m^α changes the phase of each expression only by a phase corresponding to the spin weight of that expression. The detailed verification is as follows. The operators and spin coefficients associated with the bases $\{\mathbf{e}_\mu^\pm\}$ are related to the corresponding objects of the Kinnersley basis by

$$D^\pm = D, \quad \Delta^\pm = \Delta, \quad \delta^\pm = e^{\pm i\phi} \delta, \quad \bar{\delta}^\pm = e^{\mp i\phi} \bar{\delta} \quad (\text{A1a})$$

$$\beta^\pm = -\alpha^\pm = \beta - \frac{1}{2\sqrt{2}r \sin \theta} e^{i\phi}, \quad \gamma^\pm = \gamma, \quad \rho^\pm = \rho, \quad \mu^\pm = \mu. \quad (\text{A1b})$$

For a quantity η having spin-weight 2, we have $\eta^\pm = e^{\pm 2i\phi} \eta$, and from Eqs. (A1) above we obtain the relations

$$(D + n\rho)^\pm \eta^\pm = e^{\pm 2i\phi} (D + n\rho) \eta, \quad \text{any } n, \quad (\text{A2a})$$

$$(\Delta + 2\gamma + \mu)^\pm \eta^\pm = e^{\pm 2i\phi} (\Delta + 2\gamma + \mu) \eta, \quad (\text{A2b})$$

$$(\bar{\delta} + 4\beta)^\pm \eta^\pm = e^{\pm i\phi} (\bar{\delta} + 4\beta) \eta, \quad (\text{A2c})$$

Similarly, a quantity η of spin-weight 1 satisfies the relations

$$D^\pm \eta^\pm = e^{\pm i\phi} D \eta, \quad (\text{A3a})$$

$$(\bar{\delta} + 2\beta)^\pm \eta^\pm = (\bar{\delta} + 2\beta) \eta, \quad (\text{A3b})$$

$$(\delta - 2\beta)^\pm \eta^\pm = e^{\pm 2i\phi} (\delta - 2\beta) \eta. \quad (\text{A3c})$$

From the form of the Teukolsky spin-2 and spin-1 operators of Eqs. (119) and (62),

$$\mathcal{T}_{s=2} = (\Delta + 2\gamma + \mu)(D + 3\rho) - (\delta - 2\beta)(\bar{\delta} + 4\beta) - 3\Psi_2, \quad (\text{A4})$$

$$\mathcal{T}_{s=1} = (\Delta + \mu)(D + \rho) - \delta(\bar{\delta} + 2\beta), \quad (\text{A5})$$

and Eqs. (A2) and (A3), we have the claimed relations $\mathcal{T}^\pm \Psi^\pm = \mathcal{T} \Psi$, $\mathcal{T}^\pm \Phi^\pm = \mathcal{T} \Phi$.

From the form (118) of $h_{\alpha\beta}$, Eqs. (A2) imply that the components $h_{\alpha\beta}[\Psi^\pm]$ along the basis $\{\mathbf{e}_\mu^\pm\}$ are related to the components of $h_{\alpha\beta}[\Psi]$ along the Kinnersley basis by

$$h_{22}[\Psi^\pm] = h_{22}[\Psi], \quad h_{23}[\Psi^\pm] = e^{\pm i\phi} h_{23}[\Psi], \quad h_{33}[\Psi^\pm] = e^{\pm 2i\phi} h_{33}[\Psi]. \quad (\text{A6})$$

But these are just the components of $h_{\alpha\beta}[\Psi]$ in the basis $\{\mathbf{e}_\mu^\pm\}$, implying $h_{\alpha\beta}[\Psi^\pm] = h_{\alpha\beta}[\Psi]$, as claimed.

Similarly, from the form (61) for A_α , Eqs. (A3) imply $A_\alpha[\Phi^\pm] = A_\alpha[\Phi]$. \square

APPENDIX B: ϕ_2 AND ψ_4 IN CLOSED FORM FOR STATIC PARTICLES.

We first show that the expression (50) for ϕ_2 as a series has the sum (51) and then turn to the gravitational case, showing that the series (109) has the sum (110).

Noting that the sum (50) is axisymmetric and using Eq. (18), we can write the sum in the form,

$$\phi_2 = \frac{4\pi e(\Delta\Delta_0)^{1/2}}{2\sqrt{2}Mr^2r_0} \sum_{l=1}^{\infty} \frac{1}{[l(l+1)]^{1/2}} P_l^1\left(\frac{\mathbf{r}_{\leq}}{M}\right) Q_l^1\left(\frac{\mathbf{r}_{\geq}}{M}\right) {}_{-1}Y_{l0}(\theta, \phi) Y_{l0}^*(0, 0) \quad (\text{B1})$$

$$= -\frac{4\pi e(\Delta\Delta_0)^{1/2}}{2\sqrt{2}Mr^2r_0} \bar{\delta} \sum_{l=1}^{\infty} \frac{1}{l(l+1)} P_l^1\left(\frac{\mathbf{r}_{\leq}}{M}\right) Q_l^1\left(\frac{\mathbf{r}_{\geq}}{M}\right) Y_{l0}(\theta, \phi) Y_{l0}^*(0, 0) \quad (\text{B2})$$

$$= -\frac{e(\Delta\Delta_0)^{1/2}}{2\sqrt{2}Mr^2r_0} \bar{\delta} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} P_l^1\left(\frac{\mathbf{r}_{\leq}}{M}\right) Q_l^1\left(\frac{\mathbf{r}_{\geq}}{M}\right) P_l(\cos\theta), \quad (\text{B3})$$

To sum the series (B3), we will use the relation

$$\frac{1}{\sqrt{a^2 + b^2 - 2abx - (1 - x^2)}} = \sum_{l=0}^{\infty} (2l+1) P_l(a) Q_l(b) P_l(x) =: S, \quad b \geq a \geq 1, \quad (\text{B4})$$

which follows immediately from the series expression for the Laplacian Green's function, $1/R$, written in prolate spheroidal coordinates [16]. That is, the distance R between points with prolate spheroidal coordinates (r, θ, ϕ) and (r_1, θ_1, ϕ_1) is given by

$$R^2 = r^2 + r_1^2 - 2rr_1 \cos\theta \cos\theta_1 - 2\sqrt{r^2 - \varepsilon^2} \sqrt{r_1^2 - \varepsilon^2} \sin\theta \sin\theta_1 \cos(\phi - \phi_1) - \varepsilon^2(\sin^2\theta + \sin^2\theta_1), \quad (\text{B5})$$

where $r_1, r > \varepsilon$; and the Green's function is given for $r_1 > r$ by the series

$$\begin{aligned} \frac{1}{R} = \frac{1}{\varepsilon} & \left[\sum_{l=0}^{\infty} (2l+1) \left[P_l\left(\frac{r}{\varepsilon}\right) Q_l\left(\frac{r_1}{\varepsilon}\right) P_l(\cos\theta) P_l(\cos\theta_1) + \right. \right. \\ & \left. \left. 2 \sum_{m=1}^{\infty} (-1)^m \left\{ \frac{(l-m)!}{(l+m)!} \right\}^2 P_l^m\left(\frac{r}{\varepsilon}\right) Q_l^m\left(\frac{r_1}{\varepsilon}\right) P_l^m(\cos\theta) P_l^m(\cos\theta_1) \cos m(\phi - \phi_1) \right] \right]. \quad (\text{B6}) \end{aligned}$$

Setting $\theta_1 = 0$, $\frac{r}{\varepsilon} = a$, $\frac{r_1}{\varepsilon} = b$, and $x = \cos\theta$ in Eqs. (B5) and (B6), we obtain Eq. (B4).

We next use the standard identities [10],

$$(2l+1)x \begin{bmatrix} P_l(x) \\ Q_l(x) \end{bmatrix} = (l+1) \begin{bmatrix} P_{l+1}(x) \\ Q_{l+1}(x) \end{bmatrix} + l \begin{bmatrix} P_{l-1}(x) \\ Q_{l-1}(x) \end{bmatrix}, \quad (\text{B7a})$$

$$(x^2 - 1) \frac{d}{dx} \begin{bmatrix} P_l(x) \\ Q_l(x) \end{bmatrix} = (l+1) \left\{ \begin{bmatrix} P_{l+1}(x) \\ Q_{l+1}(x) \end{bmatrix} - x \begin{bmatrix} P_l(x) \\ Q_l(x) \end{bmatrix} \right\}, \quad (\text{B7b})$$

to show the relation

$$S_1 := (-ab + x)S + a \quad (\text{B8a})$$

$$= (a^2 - 1)^{1/2}(b^2 - 1)^{1/2} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} P_l^1(a) Q_l^1(b) P_l(x). \quad (\text{B8b})$$

First, from the definition (B4) of S and Eq. (B7a), we obtain

$$xS = \sum_{l=0}^{\infty} [lP_{l-1}(a)Q_{l-1}(b) + (l+1)P_{l+1}(a)Q_{l+1}(b)] P_l(x). \quad (\text{B9})$$

Now, substituting Eq. (B9) in Eq. (B8a), we have

$$\begin{aligned} S_1 &= - \sum_{l=0}^{\infty} [(2l+1)aP_l(a)bQ_l(b) - lP_{l-1}(a)Q_{l-1}(b) - (l+1)P_{l+1}(a)Q_{l+1}(b)] P_l(x) + a \\ &= - \sum_{l=1}^{\infty} [(2l+1)aP_l(a)bQ_l(b) - lP_{l-1}(a)Q_{l-1}(b) - (l+1)P_{l+1}(a)Q_{l+1}(b)] P_l(x), \end{aligned} \quad (\text{B10})$$

where the relations $P_0 = 1$, $P_1(a) = a$, $Q_1(b) = bQ_0(b) - 1$, are used to eliminate the $l = 0$ terms. Again from Eq. (B7a), we obtain

$$\begin{aligned} S_1 &= - \sum_{l=1}^{\infty} \left\{ (2l+1) \left[\frac{(l+1)}{(2l+1)} P_{l+1}(a) + \frac{l}{(2l+1)} P_{l-1}(a) \right] \left[\frac{(l+1)}{(2l+1)} Q_{l+1}(b) + \frac{l}{(2l+1)} Q_{l-1}(b) \right] \right. \\ &\quad \left. - lP_{l-1}(a)Q_{l-1}(b) - (l+1)P_{l+1}(a)Q_{l+1}(b) \right\} P_l(x) \\ &= \sum_{l=1}^{\infty} \frac{l(l+1)}{2l+1} [P_{l+1}(a)Q_{l+1}(b) - P_{l-1}(a)Q_{l+1}(b) - P_{l+1}(a)Q_{l-1}(b) + P_{l-1}(a)Q_{l-1}(b)] P_l(x) \\ &= \sum_{l=1}^{\infty} \frac{l(l+1)}{2l+1} [P_{l+1}(a) - P_{l-1}(a)] [Q_{l+1}(b) - Q_{l-1}(b)] P_l(x) \end{aligned} \quad (\text{B11})$$

Using the definitions,

$$P_l^m(x) = (x^2 - 1)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad Q_l^m(x) = (x^2 - 1)^{m/2} \frac{d^m}{dx^m} Q_l(x), \quad |x| \geq 1, \quad (\text{B12})$$

and the identities (B7b) and (B7a), we can write

$$P_l^1(a) = \frac{l(l+1)}{(2l+1)(a^2 - 1)^{1/2}} [P_{l+1}(a) - P_{l-1}(a)]; \quad (\text{B13})$$

Using this relation and the analogous relation for $Q_l^1(b)$ to replace the bracketed expressions in Eq. (B11) yields Eq. (B8b), as claimed.

Finally, from Eqs. (B3) and (B8b), with $a = \frac{\mathbf{r}_<}{M}, b = \frac{\mathbf{r}_>}{M}$, we obtain the closed form expression (51) for ϕ_2 :

$$\phi_2 = \frac{e(\Delta\Delta_0)^{1/2}}{2\sqrt{2}Mr^2r_0}\bar{\delta}\left[\frac{M}{(\Delta\Delta_0)^{1/2}}\frac{\mathbf{r}\mathbf{r}_0 - M^2\cos\theta}{R}\right] \quad (\text{B14})$$

$$= e\frac{\Delta_0}{2\sqrt{2}r_0}\frac{\Delta\sin\theta}{r^2R^3}. \quad (\text{B15})$$

Next, we turn to the gravitational series. Again we use axisymmetry and Eq. (18) to write the sum (109) in the form

$$\begin{aligned} \psi_4 &= -\frac{\pi\mathbf{m}}{M}\frac{\Delta\Delta_0^{1/2}}{r^4r_0}\sum_{l=2}^{\infty}\left[\frac{(l-2)!}{(l+2)!}\right]^{1/2}P_l^2\left(\frac{\mathbf{r}_<}{M}\right)Q_l^2\left(\frac{\mathbf{r}_>}{M}\right)_{-2}Y_{l0}(\theta,\phi)Y_{l0}^*(0,0) \\ &= -\frac{\pi\mathbf{m}}{M}\frac{\Delta\Delta_0^{1/2}}{r^4r_0}\bar{\delta}^2\sum_{l=2}^{\infty}\frac{(l-2)!}{(l+2)!}P_l^2\left(\frac{\mathbf{r}_<}{M}\right)Q_l^2\left(\frac{\mathbf{r}_>}{M}\right)Y_{l0}(\theta,\phi)Y_{l0}^*(0,0) \\ &= -\frac{\mathbf{m}}{4M}\frac{\Delta\Delta_0^{1/2}}{r^4r_0}\bar{\delta}^2\sum_{l=2}^{\infty}(2l+1)\frac{(l-2)!}{(l+2)!}P_l^2\left(\frac{\mathbf{r}_<}{M}\right)Q_l^2\left(\frac{\mathbf{r}_>}{M}\right)P_l(\cos\theta). \end{aligned} \quad (\text{B16})$$

In this case, we wish to show

$$\begin{aligned} S_2 &\equiv (a^2 + b^2 + a^2b^2 - 1 - 4abx + 2x^2)S - (a^2 + 1)b - a(a^2 - 3)x \\ &= (a^2 - 1)(b^2 - 1)\sum_{l=1}^{\infty}\frac{(2l+1)}{(l+2)(l+1)l(l-1)}P_l^2(a)Q_l^2(b)P_l(x). \end{aligned} \quad (\text{B17})$$

Using Eq. (B9), Eq. (B7a), and relabeling the indices, we obtain

$$\begin{aligned} x^2S &= \sum_{l=0}^{\infty}\left\{\frac{l(l-1)}{(2l-1)}P_{l-2}(a)Q_{l-2}(b) + \left[\frac{(l+1)^2}{(2l+3)} + \frac{l^2}{(2l-1)}\right]P_l(a)Q_l(b) \right. \\ &\quad \left. + \frac{(l+1)(l+2)}{(2l+3)}P_{l+2}(a)Q_{l+2}(b)\right\}P_l(x). \end{aligned} \quad (\text{B18})$$

Similarly, from Eq. (B7a), we have

$$\begin{aligned} a^2P_l(a) &= \frac{(l+1)(l+2)}{(2l+1)(2l+3)}P_{l+2}(a) + \left[\frac{(l+1)^2}{(2l+1)(2l+3)} + \frac{l^2}{(2l+1)(2l-1)}\right]P_l(a) \\ &\quad + \frac{l(l-1)}{(2l+1)(2l-1)}P_{l-2}(a), \end{aligned} \quad (\text{B19})$$

together with the corresponding expression for $b^2Q_l(b)$.

As in the electromagnetic case, the $l = 0, 1$ pieces of S_2 that do not multiply S are chosen to cancel the $l = 0, 1$ pieces of the series, so that S_2 can be rewritten as

$$S_2 = \sum_{l=2}^{\infty}(a^2 + b^2 + a^2b^2 - 1 - 4bx + 2x^2)(2l+1)P_l(a)Q_l(b)P_l(x) \quad (\text{B20})$$

$$\equiv \sum_{l=2}^{\infty}(A + Bx + 2x^2)(2l+1)P_l(a)Q_l(b)P_l(x). \quad (\text{B21})$$

We use Eq. (B19) (and the corresponding expression for $b^2 Q_l(b)$) to write the part of Eq. (B21) involving A in the form

$$\begin{aligned}
& \sum_{l=2}^{\infty} A (2l+1) P_l(a) Q_l(b) P_l(x) \\
&= \sum_{l=2}^{\infty} \left(\left\{ \frac{(l+1)(l+2)}{(2l+3)} P_{l+2}(a) + \left[\frac{(l+1)^2}{(2l+3)} + \frac{l^2}{(2l-1)} \right] P_l(a) + \frac{l(l-1)}{(2l-1)} P_{l-2}(a) \right\} Q_l(b) \right. \\
&\quad + P_l(a) \left\{ \frac{(l+1)(l+2)}{(2l+3)} Q_{l+2}(b) + \left[\frac{(l+1)^2}{(2l+3)} + \frac{l^2}{(2l-1)} \right] Q_l(b) + \frac{l(l-1)}{(2l-1)} Q_{l-2}(b) \right\} \\
&\quad + \frac{1}{2l+1} \left\{ \frac{(l+1)(l+2)}{(2l+3)} P_{l+2}(a) + \left[\frac{(l+1)^2}{(2l+3)} + \frac{l^2}{(2l-1)} \right] P_l(a) + \frac{l(l-1)}{(2l-1)} P_{l-2}(a) \right\} \\
&\quad \times \left\{ \frac{(l+1)(l+2)}{(2l+1)(2l+3)} Q_{l+2}(b) + \left[\frac{(l+1)^2}{(2l+1)(2l+3)} + \frac{l^2}{(2l+1)(2l-1)} \right] Q_l(b) \right. \\
&\quad \left. \left. + \frac{l(l-1)}{(2l+1)(2l-1)} Q_{l-2}(b) \right\} - (2l+1) P_l(a) Q_l(b) \right) P_l(x). \tag{B22}
\end{aligned}$$

We next use Eq. (B7a) to rewrite $\sum_{l=2}^{\infty} B x(2l+1) P_l(a) Q_l(b) P_l(x)$ as

$$\begin{aligned}
& \sum_{l=2}^{\infty} B x(2l+1) P_l(a) Q_l(b) P_l(x) \\
&= -4 \sum_{l=2}^{\infty} \left\{ l \left[\frac{l}{(2l-1)} P_l(a) + \frac{l-1}{2l-1} P_{l-2}(a) \right] \left[\frac{l}{(2l-1)} Q_l(b) + \frac{l-1}{2l-1} Q_{l-2}(b) \right] \right. \\
&\quad \left. + (l+1) \left[\frac{(l+2)}{(2l+3)} P_{l+2}(a) + \frac{l+1}{2l+3} P_l(a) \right] \left[\frac{(l+2)}{(2l+3)} Q_{l+2}(b) + \frac{l+1}{2l+3} Q_l(b) \right] \right\} P_l(x). \tag{B23}
\end{aligned}$$

Substituting in Eq. (B21) the expressions from Eqs. (B22), (B23), and (B18), we find

$$\begin{aligned}
S_2 &= \sum_{l=2}^{\infty} \frac{(l+2)(l+1)l(l-1)}{(2l-1)^2(2l+1)(2l+3)^2} [(2l-1)P_{l+2}(a) - 2(2l+1)P_l(a) + (2l+3)P_{l-2}(a)] \\
&\quad \times [(2l-1)Q_{l+2}(b) - 2(l+1)Q_l(b) + (2l+3)Q_{l-2}(b)] P_l(x). \tag{B24}
\end{aligned}$$

From the definitions (B12) and the identities (B7), we obtain

$$\begin{aligned}
(a^2 - 1)P_l^2(a) &= \frac{(l+2)(l+1)l(l-1)}{(2l-1)(2l+1)(2l+3)} \left[(2l-1)P_{l+2}(a) \right. \\
&\quad \left. - 2(2l+1)P_l(a) + (2l+3)P_{l-2}(a) \right], \tag{B25}
\end{aligned}$$

together with the corresponding equation for $Q_l^2(b)$. Using these relations, Eq. (B24) takes the form (B17), as claimed.

Finally, from Eqs. (B16) and (B17), with $a = \frac{\mathfrak{r}_<}{M}$, $b = \frac{\mathfrak{r}_>}{M}$ as before, we obtain the closed form expression (110) for ψ_4 :

$$\begin{aligned}\psi_4 &= -\frac{\mathfrak{m}}{4M} \frac{\Delta \Delta_0^{1/2}}{r^4 r_0} \bar{\partial}^2 \left[\frac{M}{\Delta \Delta_0} \frac{M^2(\mathfrak{r}^2 + \mathfrak{r}_0^2) + \mathfrak{r}^2 \mathfrak{r}_0^2 - M^4 - 4\mathfrak{r} \mathfrak{r}_0 M^2 \cos \theta + 2M^4 \cos^2 \theta}{R} \right] \\ &= -\mathfrak{m} \frac{3}{4} \frac{\Delta_0^{3/2}}{r_0} \frac{\Delta^2 \sin^2 \theta}{r^4 R^5}.\end{aligned}\tag{B26}$$

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